Valuations on convex bodies – the classical basic facts

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1. General valuations

The natural domain for a valuation is a lattice.

In geometry, however, the following appears more frequently.

An intersectional family is a family $\mathcal{S}$ of sets such that

$$A, B \in \mathcal{S} \Rightarrow A \cap B \in \mathcal{S}, \quad \emptyset \in \mathcal{S}.$$

Definition: A function φ from an intersectional family $\mathcal{S}$ into an abelian group is called additive or a valuation if

$$\varphi(A \cup B) + \varphi(A \cap B) = \varphi(A) + \varphi(B) \tag{1}$$

for all $A, B \in \mathcal{S}$ with $A \cup B \in \mathcal{S}$, and $\varphi(\emptyset) = 0$.

Example: $\varphi(A) = \mathbf{1}_A$ (characteristic function)

The family $U(\mathcal{S})$ of all finite unions of elements from $\mathcal{S}$ is a lattice (with $\cup$ and $\cap$).

When does a valuation φ on $\mathcal{S}$ have an extension to a valuation on $U(\mathcal{S})$?

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When does a valuation φ on $\mathcal{S}$ have an extension to a valuation on $U(\mathcal{S})$?

Necessary is the inclusion-exclusion formula

$$\varphi(A_1 \cup \cdots \cup A_m) = \sum_{r=1}^{m} (-1)^{r-1} \sum_{1 \le i_1 < \cdots < i_r \le m} \varphi(A_{i_1} \cap \cdots \cap A_{i_r}),$$

for $m \in \mathbb{N}$ and $A_1, \ldots, A_m \in U(\mathcal{S})$ (easy inductive proof).

Assumption: φ is a function from an intersectional family $\mathcal{S}$ into an abelian group, with $\varphi(\emptyset) = 0$.

Definition: φ is fully additive if the inclusion-exclusion formula holds for all $A_1, \ldots, A_m \in \mathcal{S}$ with $A_1 \cup \cdots \cup A_m \in \mathcal{S}$ and for all $m \in \mathbb{N}$.

$U^\bullet(\mathcal{S})$ denotes the $\mathbb{Z}$-module spanned by the characteristic functions of the elements of $\mathcal{S}$.

Theorem 1.1 (Groemer's (1978) first extension theorem) *The following conditions are equivalent.*

(a) φ *is fully additive.*

(b) *If*

$$n_1 \mathbf{1}_{A_1} + \cdots + n_m \mathbf{1}_{A_m} = 0$$

with $A_i \in \mathcal{S}$ *and* $n_i \in \mathbb{Z}$ $(i = 1, \ldots, m)$, *then*

$$n_1 \varphi(A_1) + \cdots + n_m \varphi(A_m) = 0.$$

(c) *The functional* $\varphi^\bullet$ *defined by* $\varphi^\bullet(\mathbf{1}_A) := \varphi(A)$ *for* $A \in \mathcal{S}$ *has a* $\mathbb{Z}$*-linear extension to the module* $U^\bullet(\mathcal{S})$.

(d) φ *has an additive extension to the lattice* $U(\mathcal{S})$.

Variation (Groemer):

If φ maps into a real vector space, the same holds with

- $\mathbb{Z}$ replaced by $\mathbb{R}$,
- $U^\bullet(\mathcal{S})$ replaced by $V(\mathcal{S})$, the real vector space spanned by the characteristic functions of the elements of $\mathcal{S}$,
- $\mathbb{Z}$-linear by $\mathbb{R}$-linear.

In this case, if φ is fully additive, then Groemer defined the φ-integral of a function $f \in V(\mathcal{S})$,

$$f = a_1 \mathbf{1}_{A_1} + \cdots + a_m \mathbf{1}_{A_m}, \qquad a_1, \ldots, a_m \in \mathbb{R},$$

by

$$\int f \, \mathrm{d}\varphi := a_1 \varphi(A_1) + \cdots + a_m \varphi(A_m).$$

2. Valuations on polytopes

Now everything in $\mathbb{R}^n$ (scalar product $\langle \cdot, \cdot \rangle$ norm $\| \cdot \|$)

$\mathcal{K}^n$ set of convex bodies in $\mathbb{R}^n$
$\mathcal{P}^n$ set of convex polytopes in $\mathbb{R}^n$

Real valuations on polytopes are closely tied up with dissections of polytopes.

Definition: A dissection of the polytope $P \in \mathcal{P}^n$ is a set $\{P_1, \dots, P_m\}$ of polytopes such that $P = \bigcup_{i=1}^m P_i$ and $\dim(P_i \cap P_j) < n$ for $i \neq j$.

Assumption: G a subgroup of the affine group of $\mathbb{R}^n$

Definition: $P, Q \in \mathcal{P}^n$ are G-equidissectable if there are dissections $\{P_1, \dots, P_m\}$ of P and $\{Q_1, \dots, Q_m\}$ of Q such that $Q_i = g_i P_i$ with suitable $g_1, \dots, g_m \in G$, $i = 1, \dots, m$.

Most considered:

G_n = group of rigid motions of $\mathbb{R}^n$
T_n = group of translations of $\mathbb{R}^n$

An old example:

Theorem (Bolyai–Gerwien, 1833/35) *Any two polygons of the same area in $\mathcal{P}^2$ are G_2-equidissectable.*

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Theorem (Bolyai–Gerwien, 1833/35) *Any two polygons of the same area in $\mathcal{P}^2$ are G_2-equidissectable.*

Hilbert's third problem (1900): Is there an analogous result in three dimensions?

Dehn's (1900) negative answer is a beautiful application of valuations.

We describe it in a modified form, to introduce some further facts about valuations.

Definition: The function φ on $\mathcal{P}^n$ is weakly additive (or a weak valuation) if for each $P \in \mathcal{P}^n$ and each hyperplane H, bounding the two closed halfspaces H^+, H^-,

$$\varphi(P) = \varphi(P \cap H^+) + \varphi(P \cap H^-) - \varphi(P \cap H).$$

Theorem 2.1 *Every weak valuation on $\mathcal{P}^n$ is fully additive.*

Definition: A valuation φ on $\mathcal{K}^n$ or $\mathcal{P}^n$ is simple if $\varphi(A) = 0$ whenever $\dim A < n$.

Definition: φ is G-invariant if $\varphi(gA) = \varphi(A)$ for all $g \in G$ and all A in the domain of φ.

Lemma 2.1 *If φ is a G-invariant simple valuation on $\mathcal{P}^n$ and if the polytopes $P, Q \in \mathcal{P}^n$ are G-equidissectable, then $\varphi(P) = \varphi(Q)$.*

Proof: By Theorem 2.1, the valuation φ has an additive extension to $U(\mathcal{P}^n)$, hence the inclusion-exclusion formula can be applied to dissections $\{P_1, \ldots, P_m\}$ of P and $\{Q_1, \ldots, Q_m\}$ of Q, satisfying $g_i P_i = Q_i$ for $g_i \in G$.

Since φ is simple, the terms $\varphi(A_{i_1} \cap \cdots \cap A_{i_r})$ with $r > 1$ vanish, and what remains is

$$\begin{aligned}
\varphi(P) &= \varphi(P_1 \cup \cdots \cup P_m) = \varphi(P_1) + \cdots + \varphi(P_m) \\
&= \varphi(g_1 P_1) + \cdots + \varphi(g_m P_m) = \varphi(g_1 P_1 \cup \cdots \cup g_m P_m) \\
&= \varphi(Q_1 \cup \cdots \cup Q_m) = \varphi(Q).
\end{aligned}$$

□

Consequence: negative answer to Hilbert's third problem

Define

$$\varphi(P) := \sum_{F \in \mathcal{F}_1(P)} V_1(F) f(\gamma(P, F)) \qquad \text{for } P \in \mathcal{P}^3,$$

where

$$f(x + y) = f(x) + f(y) \qquad \text{for } x, y \in \mathbb{R},$$

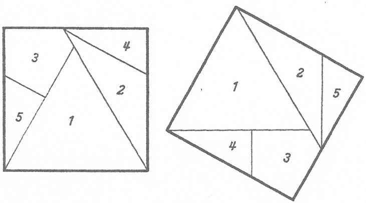
satisfying

$$f(\pi/2) = 0 \qquad f(\alpha) \neq 0,$$

for $\alpha =$ external angle of a regular tetrahedron T at one of its edges.

Then φ is weakly additive ($\Rightarrow$ fully additive) and simple, and

$$\varphi(\text{cube}) = 0 \neq \varphi(\text{regular tetrahedron}).$$



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A surprising positive result:

Theorem 2.2 (Hadwiger 1950) *Any two parallelotopes of equal volume in* $\mathbb{R}^n$ *are* T_n*-equidissectable.*

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Theorem 2.2 (Hadwiger 1950) *Any two parallelotopes of equal volume in $\mathbb{R}^n$ are T_n-equidissectable.*

A further extension theorem:

Definition: A relatively open polytope (or ro-polytope) is the relative interior of a convex polytope. $\mathcal{P}^n_{ro}$ is set of ro-polytopes. The elements of $U(\mathcal{P}^n_{ro})$ are called ro-polyhedra.

Theorem 2.3 *Any weak valuation on $\mathcal{P}^n$ has an additive extension to $U(\mathcal{P}^n_{ro})$.*

This is helpful to prove the following fundamental result.

Theorem 2.4 *Let φ be a translation invariant valuation on $\mathcal{P}^n$ with values in a rational vector space X. Then*

$$\varphi(\lambda P) = \sum_{r=0}^{n} \lambda^r \varphi_r(P) \qquad \text{for } P \in \mathcal{P}^n \text{ and rational } \lambda \geq 0.$$

Here $\varphi_r : \mathcal{P}^n \to X$ is a translation invariant valuation which is rational homogeneous of degree r ($r = 0, \ldots, n$).

Setting $\lambda = 1$, gives

$$\varphi = \varphi_0 + \cdots + \varphi_n,$$

the McMullen decomposition.

Historical note. Theorem 2.4 was already stated by Hadwiger (1945), but without proof.

His later work gives a proof of the decomposition only for simple valuations.

The question for the general result was posed by McMullen (1974), at an Oberwolfach conference.

Proofs by McMullen (1974), Meier (1977), Spiegel (1978).

Later proofs of generalizations by Pukhlikov and Khovanskii (1992), Alesker (1998).

Consequence: A polynomial expansion under Minkowski addition

Theorem 2.5 *Let $\varphi : \mathcal{P}^n \to X$ (a rational vector space) be a translation invariant valuation which is rational homogeneous of degree $m \in \{1, \ldots, n\}$. Then there is a polynomial expansion*

$$\varphi(\lambda_1 P_1 + \cdots + \lambda_k P_k)$$

$$= \sum_{r_1,\ldots,r_k=0}^{m} \binom{m}{r_1 \ldots r_k} \lambda_1^{r_1} \cdots \lambda_k^{r_k} \overline{\varphi}(\underbrace{P_1, \ldots, P_1}_{r_1}, \ldots, \underbrace{P_k, \ldots, P_k}_{r_k}),$$

for all $P_1, \ldots, P_k \in \mathcal{P}^n$ and all rational $\lambda_1, \ldots, \lambda_k \geq 0$.

Here $\overline{\varphi} : (\mathcal{P}^n)^m \to X$ is a symmetric mapping, which is translation invariant and Minkowski additive in each variable.

Representation results for translation invariant valuations:

There are representation results for weakly continuous valuations on polytopes, by Hadwiger (1952), McMullen (1983).

Two characterizations of the volume, V_n, from Hadwiger's (1957) book:

Theorem 2.6 *Let* $\varphi : \mathcal{P}^n \to \mathbb{R}$ *be a translation invariant valuation which is simple and nonnegative.*
Then $\varphi = cV_n$ *with a constant* c.

Theorem 2.7 *Let* $\varphi : \mathcal{P}^n \to \mathbb{R}$ *be a translation invariant valuation which is homogeneous of degree* n.
Then $\varphi = cV_n$ *with a constant* c.

3. Examples of valuations from convex geometry

- The identity mapping $\mathcal{K}^n \to \mathcal{K}^n$

Note that $\mathcal{K}^n$ with Minkowski addition is an abelian semigroup with cancellation law, and that

$$(K \cup L) + (K \cap L) = K + L$$

if $K, L, K \cup L \in \mathcal{K}^n$.

- A mapping φ from $\mathcal{K}^n$ into an abelian group is Minkowski additive if it satisfies

$$\varphi(K + L) = \varphi(K) + \varphi(L), \qquad K, L \in \mathcal{K}^n.$$

Every such mapping is a valuation, in fact fully additive.

- In particular, the support function $h(K, \cdot) = h_K$, defined by $h(K, u) := \max\{\langle u, x \rangle : x \in K\}$, yields a valuation.

• Mixed volume valuations

The mixed volume $V : (\mathcal{K}^n)^n \to \mathbb{R}$ is defined by

$$V_n(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1,\ldots,i_n=1}^{m} \lambda_{i_1} \cdots \lambda_{i_n} V(K_{i_1}, \ldots, K_{i_n})$$

for $m \in \mathbb{N}$, $K_1, \ldots, K_m \in \mathcal{K}^n$, $\lambda_1, \ldots, \lambda_m \geq 0$ (and symmetry).

For $p \in \{1, \ldots, n\}$ and fixed $M_{p+1}, \ldots, M_n \in \mathcal{K}^n$,

$$\varphi(K) := V(\underbrace{K, \ldots, K}_{p}, M_{p+1}, \ldots, M_n), \qquad K \in \mathcal{K}^n,$$

defines a valuation φ.

It is translation invariant, continuous, and homogeneous of degree p.

- Intrinsic volumes

Let

$$\kappa_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(1+\frac{n}{2}\right)}, \qquad \omega_n = n\kappa_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}.$$

The special mixed volume valuation

$$V_j(K) := \frac{\binom{n}{j}}{\kappa_{n-j}} V(\underbrace{K,\ldots,K}_{j},\underbrace{B^n,\ldots,B^n}_{n-j}),$$

defined by the Steiner formula

$$V_n(K+\rho B^n) = \sum_{j=0}^{n} \rho^{n-j}\kappa_{n-j}V_j(K), \qquad \rho \geq 0,$$

is the jth intrinsic volume. It is rigid motion invariant.

• Support and curvature measures

For $K \in \mathcal{K}^n$ and $x \in \mathbb{R}^n \setminus K$, let
$d(K, x)$ be the distance of x from K,
$p(K, x) \in K$ the point realizing the distance,
$u(K, x) := (x - p(K, x))/d(K, x)$, unit vector from $p(K, x)$ to x.

Let $\Sigma^n := \mathbb{R}^n \times \mathbb{S}^{n-1}$ and $\eta \in \mathcal{B}(\Sigma^n)$ ($\mathcal{B}$ = Borel sets).

The local parallel set

$$M_\rho(K, \eta) := \{x \in \mathbb{R}^n : 0 < d(K, x) \leq \rho \text{ and } (p(K, x), u(K, x)) \in \eta\}$$

has (n-dimensional Hausdorff) measure

$$\mathcal{H}^n(M_\rho(K, \eta)) = \sum_{j=0}^{n-1} \rho^{n-j} \kappa_{n-j} \Lambda_j(K, \eta) \qquad \text{for } \rho \geq 0.$$

This defines finite measures $\Lambda_0(K, \cdot), \ldots, \Lambda_{n-1}(K, \cdot)$ on Σ^n.

$\Lambda_j(K,\cdot)$ is the jth support measure of K.

The map $K \mapsto \Lambda_j(K,\cdot)$ is a valuation, with values in the vector space of finite signed Borel mesures on Σ^n.

$$C_j(K,\beta) := \frac{n\kappa_{n-j}}{\binom{n}{j}}\Lambda_j(K,\beta\times\mathbb{S}^{n-1}), \qquad \beta\in\mathcal{B}(\mathbb{R}^n),$$

defines the jth curvature measure of K, supplemented by

$$C_n(K,\beta) := \mathcal{H}^n(K\cap\beta).$$

$$S_j(K,\omega) := \frac{n\kappa_{n-j}}{\binom{n}{j}}\Lambda_j(K,\mathbb{R}^n\times\omega), \qquad \omega\subset\mathcal{B}(\mathbb{S}^{n-1}),$$

defines the jth area measure of K.

4. Continuous valuations on convex bodies

Continuity on $\mathcal{K}^n$ refers to the Hausdorff metric.

All the valuations in Sec. 3 have continuity and invariance properties.

Theorem 4.1 (Groemer's second extension theorem) *Every continuous valuation on $\mathcal{K}^n$ with values in a topological vector space has an additive extension to the lattice $U(\mathcal{K}^n)$.*

Assumption: φ translation invariant, continuous valuation on $\mathcal{K}^n$, with values in a topological vector space

Then

$$\varphi(\lambda K) = \sum_{i=0}^{n} \lambda^i \varphi_i(K) \quad \text{for } K \in \mathcal{K}^n \text{ and } \lambda \geq 0,$$

with φ_i continuous and homogeneous of degree i.

If φ is homogeneous of degree m, then

$$\varphi(\lambda_1 K_1 + \cdots + \lambda_k K_k)$$

$$= \sum_{r_1,\ldots,r_k=0}^{m} \binom{m}{r_1 \ldots r_k} \lambda_1^{r_1} \cdots \lambda_k^{r_k} \,\overline{\varphi}(\underbrace{K_1,\ldots,K_1}_{r_1},\ldots,\underbrace{K_k,\ldots,K_k}_{r_k})$$

for $K_1,\ldots,K_k \in \mathcal{K}^n$ and $\lambda_1,\ldots,\lambda_k \geq 0$.

The mapping

$$K \mapsto \overline{\varphi}(\underbrace{K,\ldots,K}_{r}, M_{r+1},\ldots,M_m), \tag{2}$$

with fixed convex bodies $M_{r+1},\ldots,M_m$, is a continuous, translation invariant valuation, homogeneous of degree r.

A second historical incentive

Consider the kinematic integral

$$\psi(K,M) := \int_{G_n} \chi(K \cap gM)\, \mu(\mathrm{d}g)$$

for $K, M \in \mathcal{K}^n$, where χ is the Euler characteristic and μ is the Haar measure on G_n.

The result is

$$\psi(K,M) = \sum_{i,j=0}^{n} c_{ij} V_i(K) V_j(M).$$

Blaschke (1939) noticed that in his proof (of a special case) the valuation property of the V_j played an important role.

He suggested, therefore, that the intrinsic volumes might be characterized by additivity, rigid motion invariance, and local boundedness. He did not succeed with a proof.

In fact, on $\mathcal{K}^n$, local boundedness is not a suitable assumption.

However, with continuity instead, Hadwiger (1951/52) succeeded:

Theorem 4.2 (Hadwiger's characterization theorem)
If $\varphi : \mathcal{K}^n \to \mathbb{R}$ is a continuous, rigid motion invariant valuation, then there are constants $c_0, \ldots, c_n$ such that

$$\varphi(K) = \sum_{j=0}^{n} c_j V_j(K)$$

for all $K \in \mathcal{K}^n$.

A slightly simplified version of his proof: Chen (2004)

A considerably shorter, elegant proof: Klain (1995)

Hadwiger (1950, 1956) made many integral-geometric applications of his characterization.

Example: Consider

$$\psi(K, M) := \int_{G_n} \chi(K \cap gM)\,\mu(\mathrm{d}g).$$

For fixed K, the function $\psi(K, \cdot)$ satifies the assumptions of Theorem 4.2, hence

$$\psi(K, M) = \sum_{j=0}^{n} c_j(K) V_j(M).$$

Repeat the argument with variable K, to obtain that

$$\psi(K, M) = \sum_{j=0}^{n} c_{ij} V_i(K) V_j(M).$$

The constants c_{ij} can be determined by applying the formula to balls of different radii.

Translation invariant valuations

Val the real vector space of translation invariant, continuous real valuations on $\mathcal{K}^n$

Val$_m$ the subspace of valuations that are homogeneous of degree m

Val$_m^+$ the subset of even valuations

Val$_m^-$ the subset of odd valuations

φ is even if $\varphi(-K) = \varphi(K)$, and odd if $\varphi(-K) = -\varphi(K)$, for all K in the domain of φ.

Then, by the McMullen decomposition (and trivially)

$$\mathbf{Val} = \bigoplus_{m=0}^{n} \mathbf{Val}_m, \qquad \mathbf{Val}_m = \mathbf{Val}_m^+ \oplus \mathbf{Val}_m^-.$$

Some classification results concerning Val

$\mathbf{Val}_0$ is spanned by the Euler characteristic,

$\mathbf{Val}_n$ is spanned by the volume.

Theorem 4.3 (McMullen 1980) *Each* $\varphi \in \mathbf{Val}_{n-1}$ *has a representation*

$$\varphi(K) = \int_{\mathbb{S}^{n-1}} f(u)\, S_{n-1}(K, \mathrm{d}u) \qquad \text{for } K \in \mathcal{K}^n,$$

with a continuous function $f : \mathbb{S}^{n-1} \to \mathbb{R}$. *This function is uniquely determined up to adding the restriction of a linear function.*

Theorem 4.4 (Klain 1995) *If* $\varphi \in \mathbf{Val}^+$ *is simple, then* $\varphi(K) = cV_n(K)$ *for* $K \in \mathcal{K}^n$, *with some constant* c.

Theorem 4.5 (R.S. 1996) *If* $\varphi \in \mathbf{Val}^-$ *is simple, then*

$$\varphi(K) = \int_{\mathbb{S}^{n-1}} g(u)\, S_{n-1}(K, \mathrm{d}u) \qquad \textit{for } K \in \mathcal{K}^n,$$

with an odd continuous function $g : \mathbb{S}^{n-1} \to \mathbb{R}$.

Theorem 4.5 (R.S. 1996) *If $\varphi \in \mathbf{Val}^-$ is simple, then*

$$\varphi(K) = \int_{\mathbb{S}^{n-1}} g(u)\, S_{n-1}(K, \mathrm{d}u) \qquad \text{for } K \in \mathcal{K}^n,$$

with an odd continuous function $g : \mathbb{S}^{n-1} \to \mathbb{R}$.

Klain's volume characterization (Theorem 4.4) has a useful consequence:

$G(n, m)$ the Grassmannian

If $\varphi \in \mathbf{Val}_m$ and $L \in G(n, m)$, then $\varphi(K) = c_\varphi(L) V_m(K)$ for $K \subset L$, with a real constant $c_\varphi(L)$. This defines the (continuous) Klain function $c_\varphi : G(n, m) \to \mathbb{R}$.

Theorem 4.6 *A valuation in $\mathbf{Val}_m^+$ ($m \in \{1, \dots, n-1\}$) is uniquely determined by its Klain function.*

5. Measure-valued valuations

Recall that the support measures on $\Sigma^n := \mathbb{R}^n \times \mathbb{S}^{n-1}$,

$$\Lambda_0(K,\cdot),\dots,\Lambda_{n-1}(K,\cdot),$$

are defined by

$$\mathcal{H}^n(M_\rho(K,\eta)) = \sum_{j=0}^{n-1} \rho^{n-j}\kappa_{n-j}\Lambda_j(K,\eta).$$

As functions of K, they are valuations and weakly continuous.

The latter means that $K_i \to K$ in the Hausdorff metric implies

$$\lim_{i\to\infty} \int_{\Sigma^n} f \,\mathrm{d}\Lambda_m(K_i,\cdot) = \int_{\Sigma^n} f \,\mathrm{d}\Lambda_m(K,\cdot)$$

for every continuous function $f : \Sigma^n \to \mathbb{R}$.

The measure $\Lambda_j(K, \cdot)$ is concentrated on the normal bundle $\mathrm{Nor}\, K$ of K.

Valuation property and weak continuity carry over to the mappings C_j and S_j.

The curvature measure $C_j(K, \cdot)$ is a Borel measure on $\mathbb{R}^n$, concentrated on $\mathrm{bd}\, K$ for $j \leq n - 1$, and on K for $j = n$.

The area measure $S_j(K, \cdot)$ is a Borel measure on the unit sphere $\mathbb{S}^{n-1}$.

Behaviour under the motion group:

For $g \in G_n$, let g_0 be the rotation part (that is, $gx = g_0 x + t$ for $x \in \mathbb{R}^n$). Define

$$
\begin{aligned}
g\eta &:= \{(gx, g_0 u) : (x, u) \in \eta\} \quad \text{for } \eta \subset \Sigma^n, \\
g\beta &:= \{gx : x \in \beta\} \quad \text{for } \beta \subset \mathbb{R}^n, \\
g\omega &:= \{g_0 u : u \in \omega\} \quad \text{for } \omega \subset \mathbb{S}^{n-1}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\Lambda_j(gK, g\eta) &= \Lambda_j(K, \eta), \\
C_j(gK, g\beta) &= C_j(K, \beta) \\
S_j(gK, g\omega) &= S_j(K, \omega).
\end{aligned}
$$

In each case, we call this rigid motion equivariance.

Characterization theorems à la Hadwiger:

Theorem 5.1 (R.S. 1978) *Let φ be a map from $\mathcal{K}^n$ into the set of finite Borel measures on $\mathbb{R}^n$, such that:*

(a) φ *is a valuation,*

(b) φ *is rigid motion equivariant,*

(c) φ *is weakly continuous,*

(d) φ *is locally determined, which means: if $\beta \subset \mathbb{R}^n$ is open and $K \cap \beta = L \cap \beta$, then $\varphi(K, \beta') = \varphi(L, \beta')$ for every Borel set $\beta' \subset \beta$.*

Then

$$\varphi(K, \beta) = \sum_{i=0}^{n} c_i C_i(K, \beta)$$

with $c_0, \dots, c_n \geq 0$, for $K \in \mathcal{K}^n$ and $\beta \in \mathcal{B}(\mathbb{R}^n)$.

Theorem 5.2 (R.S. 1975) *Let φ be a map from $\mathcal{K}^n$ into the set of finite signed Borel measures on $\mathbb{S}^{n-1}$, such that*

(a) φ *is a valuation,*

(b) φ *is rigid motion equivariant,*

(c) φ *is weakly continuous,*

(d) φ *is locally determined, which means: if $\omega \subset \mathbb{S}^{n-1}$ is a Borel set and if $\tau(K, \omega) = \tau(L, \omega)$, then $\varphi(K, \omega) = \varphi(L, \omega)$.*

Then

$$\varphi(K, \omega) = \sum_{i=0}^{n-1} c_i S_i(K, \omega)$$

with real constants $c_0, \dots, c_{n-1}$, for $K \in \mathcal{K}^n$ and $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$.

$\tau(K, \omega)$ is the inverse spherical image of K at ω.

Theorem 5.3 (Glasauer 1997) *Let φ be a map from $\mathcal{P}^n$ into the set of finite signed Borel measures on Σ^n, such that:*

(a) φ *is rigid motion equivariant,*

(b) φ *is locally determined, which means: if $\eta \in \mathcal{B}(\Sigma^n)$ and $K, L \in \mathcal{K}^n$ satisfy $\eta \cap \mathrm{Nor}\, K = \eta \cap \mathrm{Nor}\, L$, then $\varphi(K, \eta) = \varphi(L, \eta)$.*

Then

$$\varphi(K, \eta) = \sum_{j=0}^{n-1} c_j \Lambda_j(K, \eta)$$

with real constants $c_0, \ldots, c_{n-1}$, for $K \in \mathcal{K}^n$ and $\eta \in \mathcal{B}(\Sigma^n)$.

Note that the valuation property need not be assumed here.

6. Tensor-valued valuations

Recall that the intrinsic volumes were derived from the volume,

$$V_n(K+\rho B^n)=\sum_{j=0}^{n}\rho^{n-j}\kappa_{n-j}V_j(K).$$

It is natural to replace the volume

$$V_n(K)=\int_K \mathrm{d}x$$

by the moment vector

$$\int_K x\,\mathrm{d}x$$

or by higher moments, for example

$$\int_K \langle x,e_i\rangle\langle x,e_j\rangle\,\mathrm{d}x,$$

that is, by tensor-valued functionals.

Conventions about tensors

We use the scalar product $\langle \cdot, \cdot \rangle$ to identify $\mathbb{R}^n$ with its dual space. Hence, we don't distinguish between covariant and contravariant tensors.

Thus, an r-tensor, or tensor of rank r, on $\mathbb{R}^n$ is an r-linear mapping from $(\mathbb{R}^n)^r$ to $\mathbb{R}$.

It is *symmetric* if it is independent under permutation of its arguments.

Let $\mathbb{T}^r$ denote the real vector space (with its standard topology) of symmetric r-tensors on $\mathbb{R}^n$.

By definition, $\mathbb{T}^0 = \mathbb{R}$, and by identification, $\mathbb{T}^1 = \mathbb{R}^n$.

The symmetric tensor product of $a \in \mathbb{T}^r$ and $b \in \mathbb{T}^s$ is defined by

$$(a \odot b)(x_1, \ldots, x_{r+s})$$
$$:= \frac{1}{(r+s)!} \sum_{\sigma \in \mathcal{S}(r+s)} a(x_{\sigma(1)}, \ldots, x_{\sigma(r)}) b(x_{\sigma(r+1)}, \ldots, x_{\sigma(r+s)}),$$

where $\mathcal{S}(k)$ denotes the group of permutations of the numbers $1, \ldots, k$.

Thus $a \odot b \in \mathbb{T}^{r+s}$.

The symmetric tensor product extends in an obvious way to more than two factors.

Abbreviations:

$$a \odot b =: ab, \qquad \underbrace{a \odot \cdots \odot a}_{r} =: a^r, \qquad a^0 := 1.$$

Example: for a vector $a \in \mathbb{R}^n$, the r-tensor a^r is given by

$$a^r(x_1, \ldots, x_r) = \langle a, x_1 \rangle \cdots \langle a, x_r \rangle, \qquad x_1, \ldots, x_r \in \mathbb{R}^n.$$

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Let $(e_1, \dots, e_n)$ be an orthonormal basis of $\mathbb{R}^n$. For an r-tensor $T \in \mathbb{T}^r$, let

$$t_{i_1 \dots i_r} := T(e_{i_1}, \dots, e_{i_r}).$$

Then

$$T = \sum_{1 \le i_1 \le \cdots \le i_r \le n} t_{i_1 \dots i_r} e_{i_1} \cdots e_{i_r}$$

is the coordinate representation of T.

Moment and Minkowski tensors

Definition: 6.1 For $r \in \mathbb{N}_0$, the rth moment tensor is defined by

$$\Psi_r(K) := \frac{1}{r!} \int_K x^r \, \mathrm{d}x, \qquad K \in \mathcal{K}^n.$$

Thus, $\Psi_r(K) \in \mathbb{T}^r$. Explicitly

$$\Psi_r(K)(y_1, \ldots, y_r) = \frac{1}{r!} \int_K \langle x, y_1 \rangle \cdots \langle x, y_r \rangle \, \mathrm{d}x$$

for $y_1, \ldots, y_r \in \mathbb{R}^n$.

$\Psi_r : \mathcal{K}^n \to \mathbb{T}^r$ is a simple valuation.

The factor $1/r!$ is for convenience.

Translation behaviour:

$$\Psi_r(K+t) = \sum_{j=0}^{r} \frac{1}{j!} \Psi_{r-j}(K) t^j.$$

This is called polynomial behaviour, but remember that

$$\Psi_{r-j}(K) t^j = \Psi_{r-j}(K) \odot \underbrace{t \odot \cdots \odot t}_{j}.$$

Rotation behaviour: For $\vartheta \in O(n)$,

$$\Psi_r(\vartheta K) = \vartheta \Psi_r(K),$$

where the operation of $O(n)$ on $\mathbb{T}^r$ is defined by

$$(\vartheta a)(y_1, \dots, y_r) = a(\vartheta^{-1} y_1, \dots, \vartheta^{-1} y_r)$$

for $a \in \mathbb{T}^r$.

Definition: 6.2 The Minkowski tensors are defined by

$$\Phi_k^{r,s}(K) := \frac{1}{r!s!}\frac{\omega_{n-k}}{\omega_{n-k+s}}\int_{\Sigma^n} x^r u^s \, \Lambda_k(K, \mathrm{d}(x,u))$$

for $k = 1, \ldots, n-1$ and $r, s \in \mathbb{N}_0$. Further,

$$\Phi_n^{r,0}(K) := \Psi_r(K).$$

Again, the normalizing factors are for convenience.

The definition

$$\Phi_k^{r,s} := 0 \quad \text{if } k \notin \{0, \ldots, n\} \text{ or } r \notin \mathbb{N}_0 \text{ or } s \notin \mathbb{N}_0 \text{ or } k = n,\ s \neq 0$$

will allow us to extend some summations formally over all nonnegative integers.

Now we can formulate a Steiner-type formula.

Theorem 6.1 *For $r \in \mathbb{N}_0$, $K \in \mathcal{K}^n$ and $\rho \geq 0$,*

$$\Psi_r(K + \rho B^n) = \sum_{k=0}^{n+r} \rho^{n+r-k} \kappa_{n+r-k} V_k^{(r)}(K),$$

where

$$V_k^{(r)} = \sum_{s \in \mathbb{N}_0} \Phi_{k-r+s}^{r-s,s}.$$

For $k = 0$, this reduces to the ordinary Steiner formula.