Topics in geometric inference
Lecture I: Voronoi covariance measure

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Joint works with F. Chazal, D. Cohen-Steiner, L. Cuel, L. Guibas, J.O Lachaud, M. Ovsjanikov and B. Thibert

Workshop on Tensor Valuations in Stochastic Geometry and Imaging

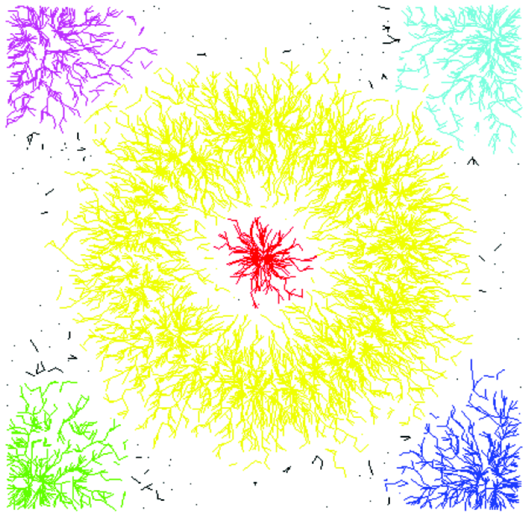
21–26 September 2014, Sandbjerg Estate, Sønderborg, Denmark

Geometric inference

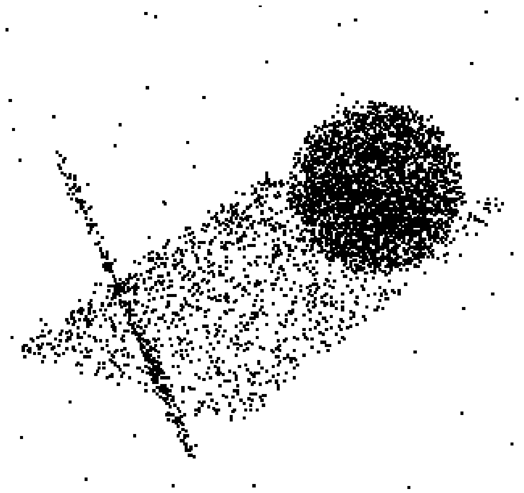
Given: — An unknown object K (compact set) in $\mathbb{R}^d$

— A finite point set $P \subseteq \mathbb{R}^d$ approximating K.

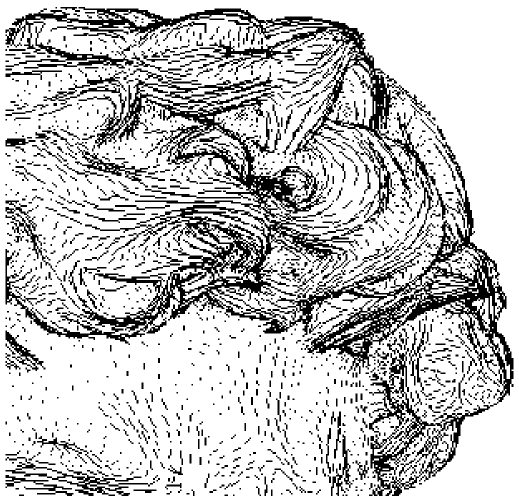
What amount of the topology and geometry of K can we recover from P ?



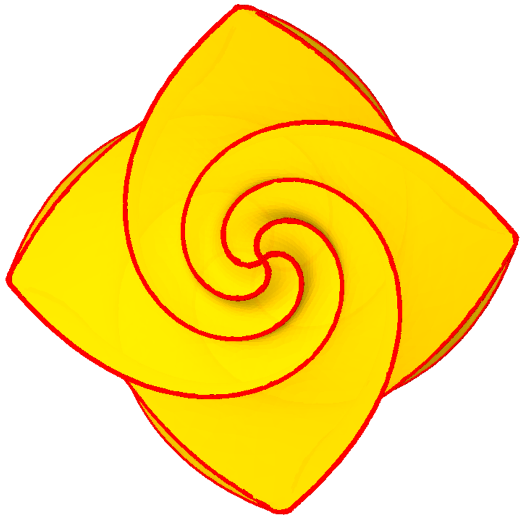
$\#$ connected components



intrinsic dimension



curvature

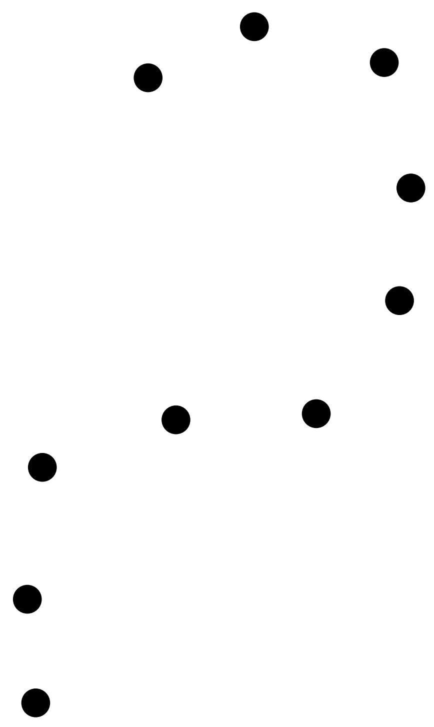


location of sharp features

1. Tube formulas, curvature measures and their stability
2. Voronoi covariance measure
3. Distance to a measure and generalized VCM
4. Computations

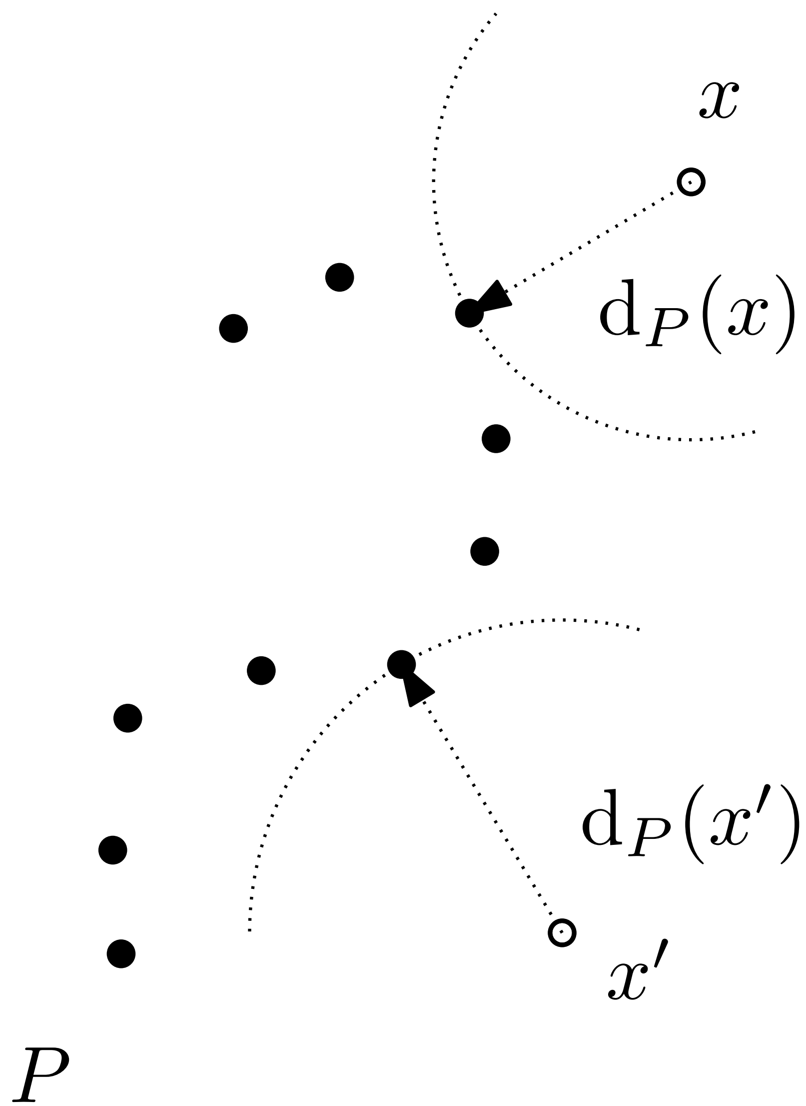
1. Tube formulas and curvature measures

Distance function and offsets



Distance function: $\mathrm{d}_P : x \in \mathbb{R}^d \mapsto \min_{p \in P} \|x - p\|$

Distance function and offsets



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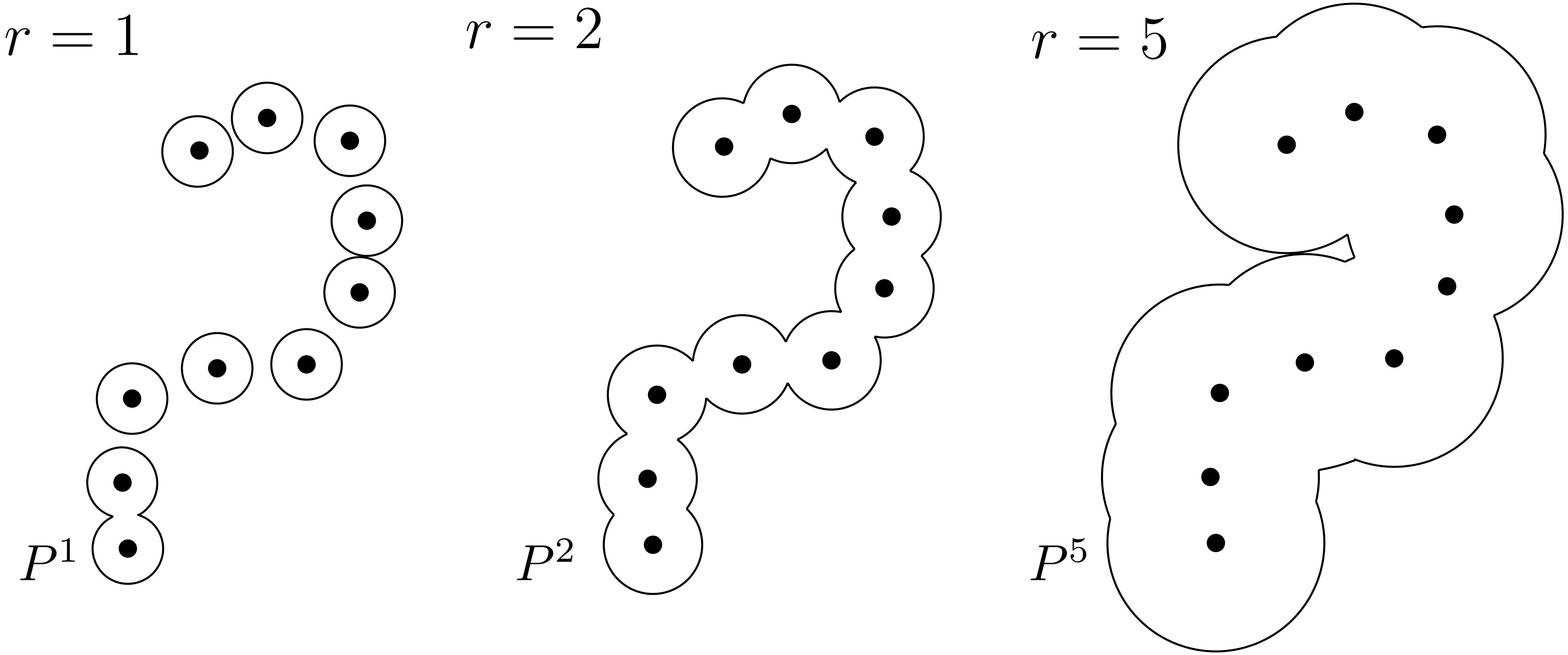

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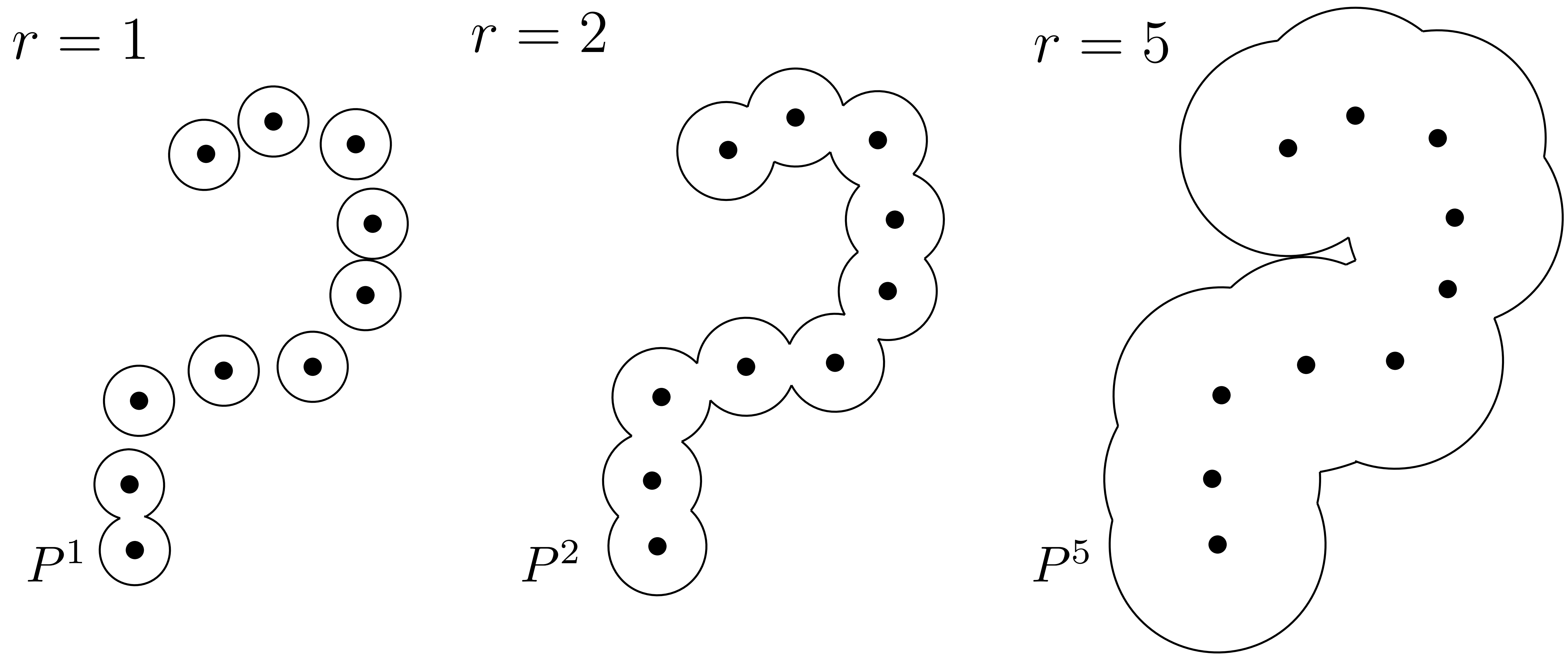

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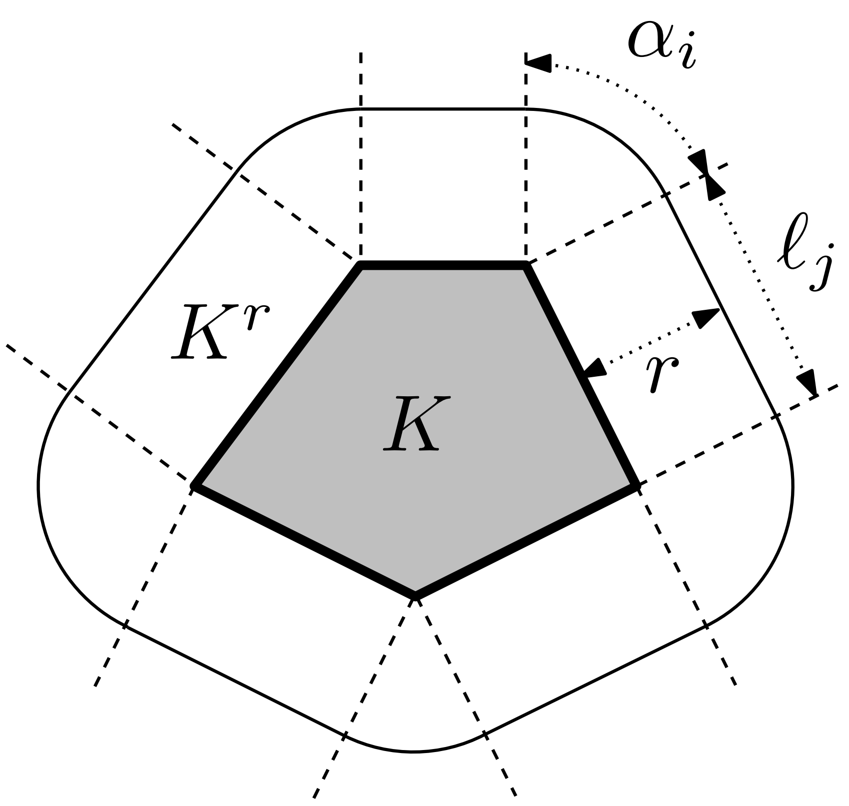
Hausdorff distance: $\mathrm{d_H}(P, K) := \|\, \mathrm{d}_P - \mathrm{d}_K \,\|_\infty.$

Tube formulas and curvature



Theorem (Steiner-Minkowski): For every compact convex subset K of $\mathbb{R}^d$, the function $r \mapsto \mathcal{H}^{\mathrm{d}}(K^r)$ is a degree d polynomial.

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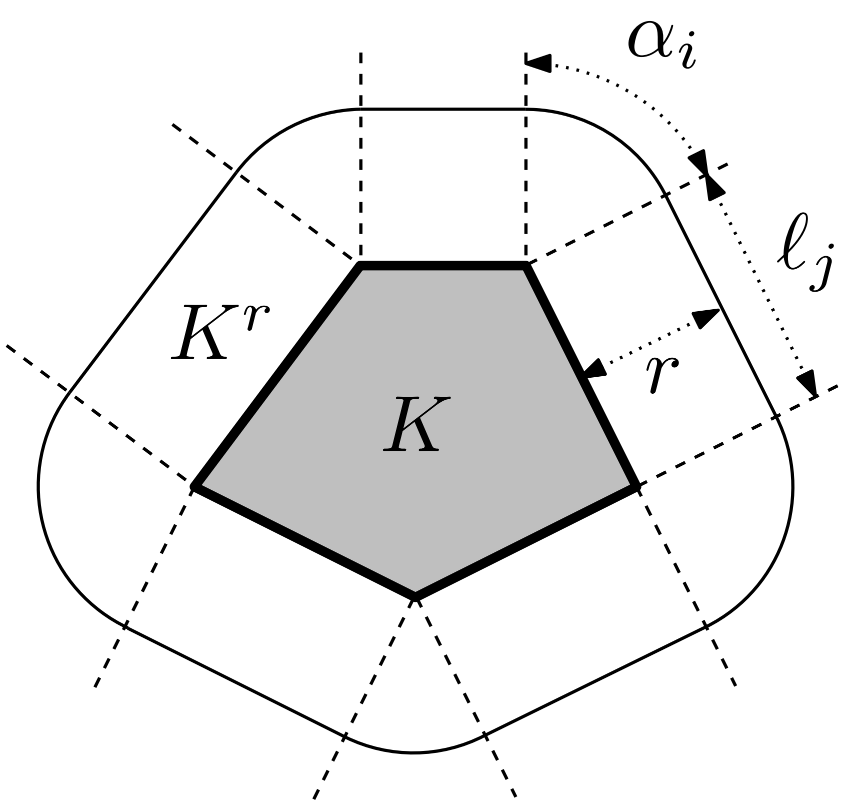

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$$\mathcal{H}^2(P^r) = \mathcal{H}^2(P) + r\mathcal{H}^1(\partial P) + r^2\pi$$

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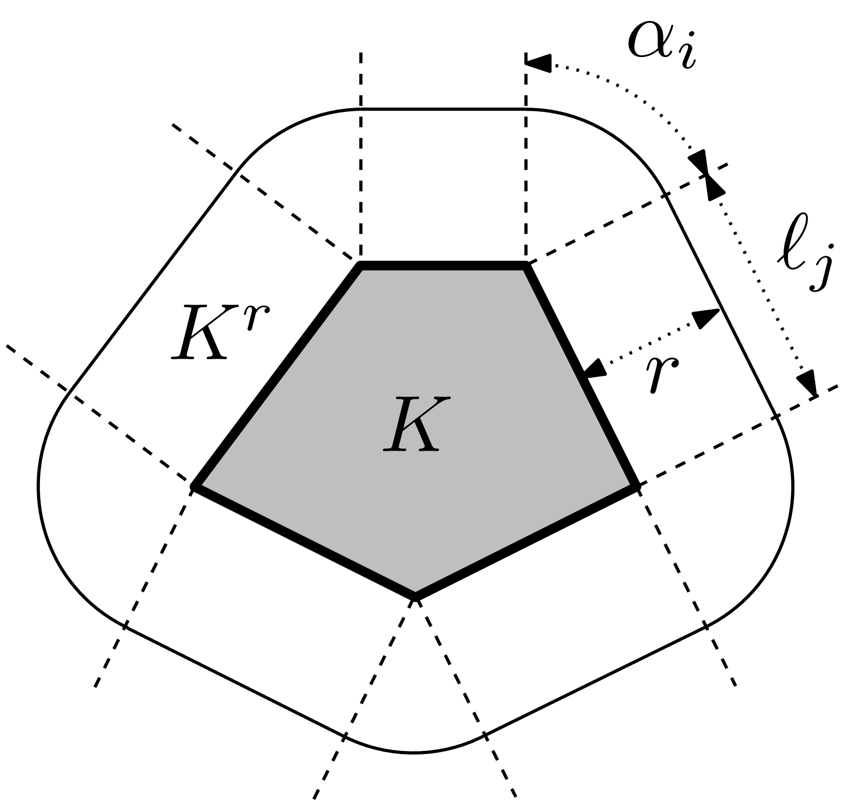

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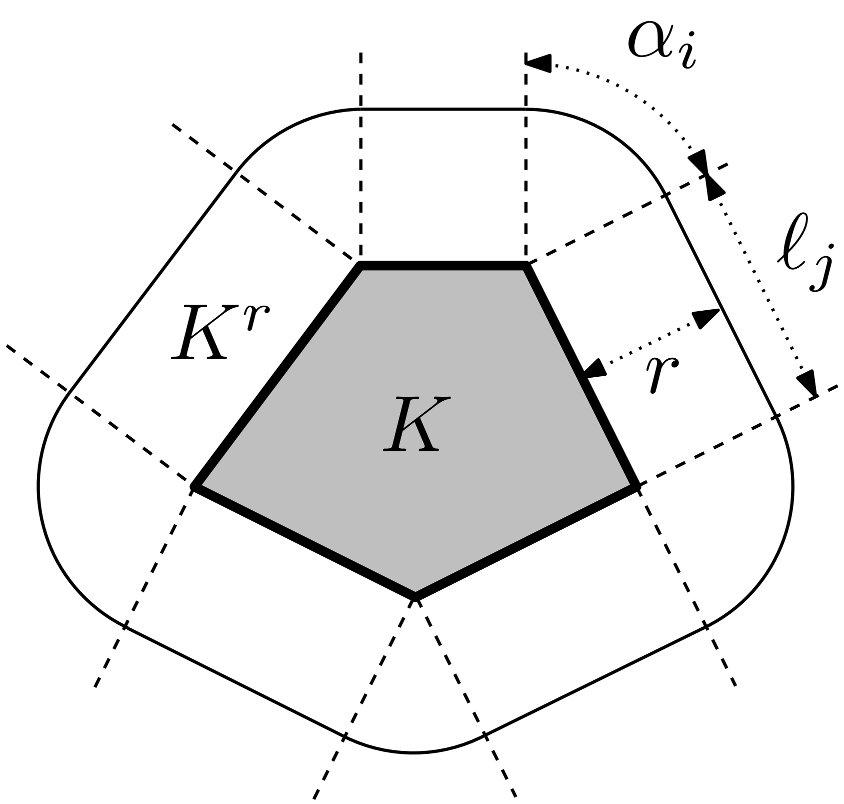

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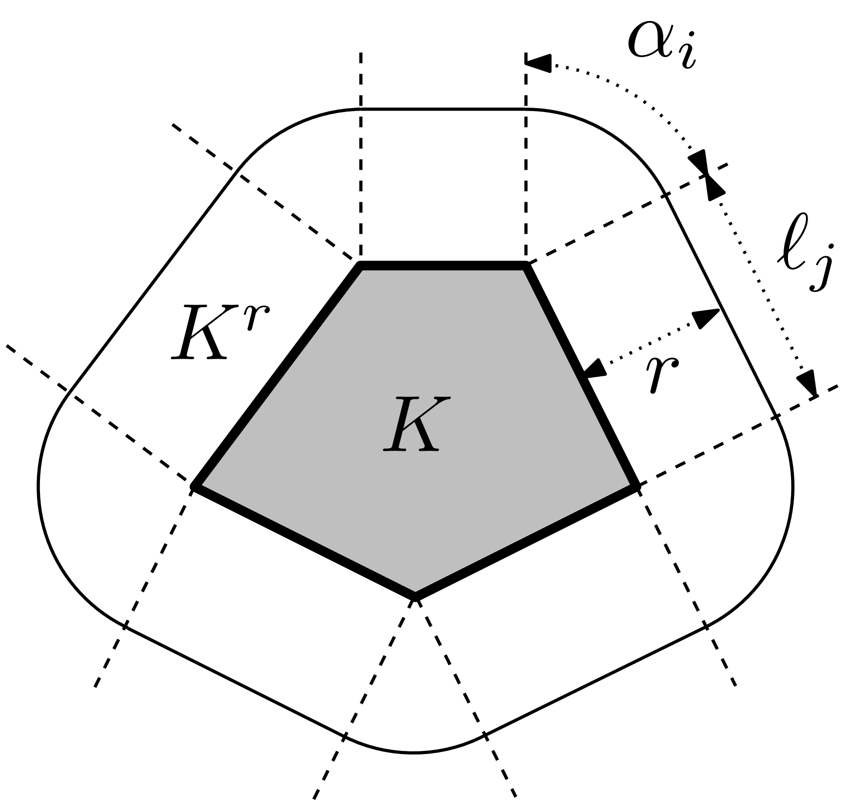

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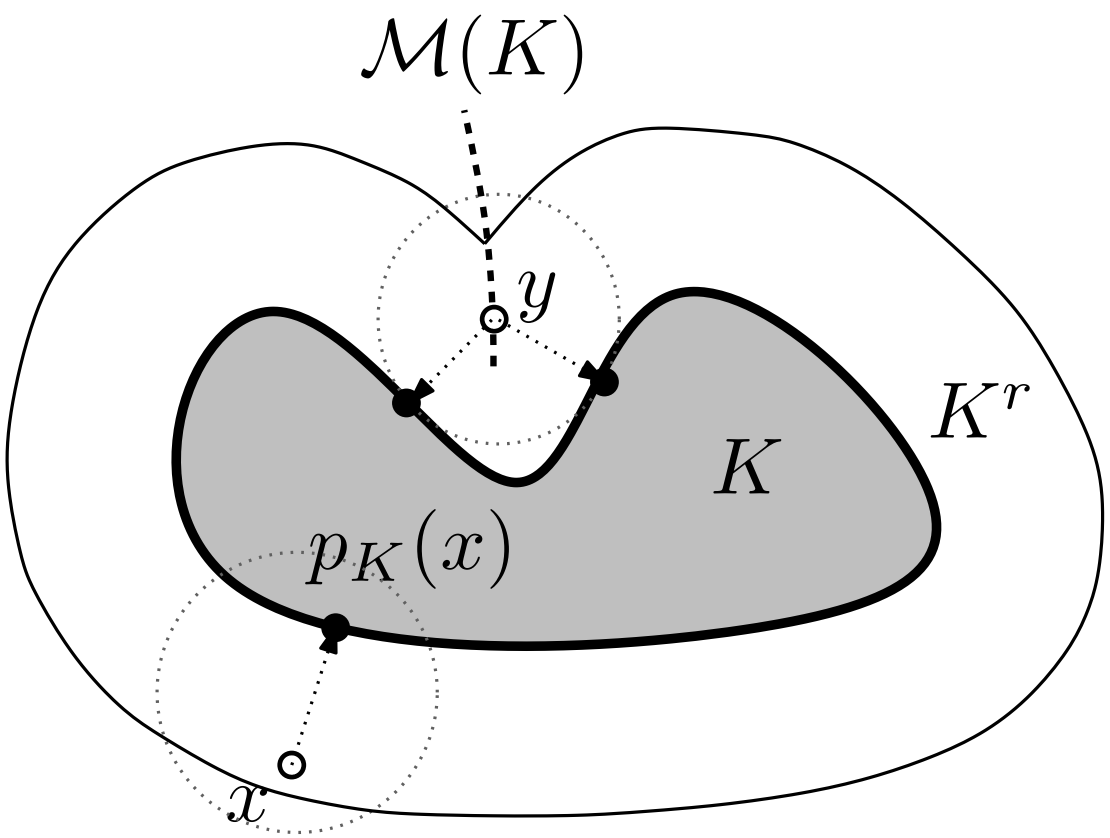
$\Phi_K^2 = \mathrm{area}(S)$

$\Phi_K^1 =$ tot. mean curvature

$\Phi_K^0 =$ tot. Gaussian curvature

Federer's local tube formula



Definition: The medial axis of $K \subseteq \mathbb{R}^d$ is

$$\mathcal{M}(K) := \{x \in \mathbb{R}^d;\ \#\operatorname{proj}_K(x) > 1\}$$

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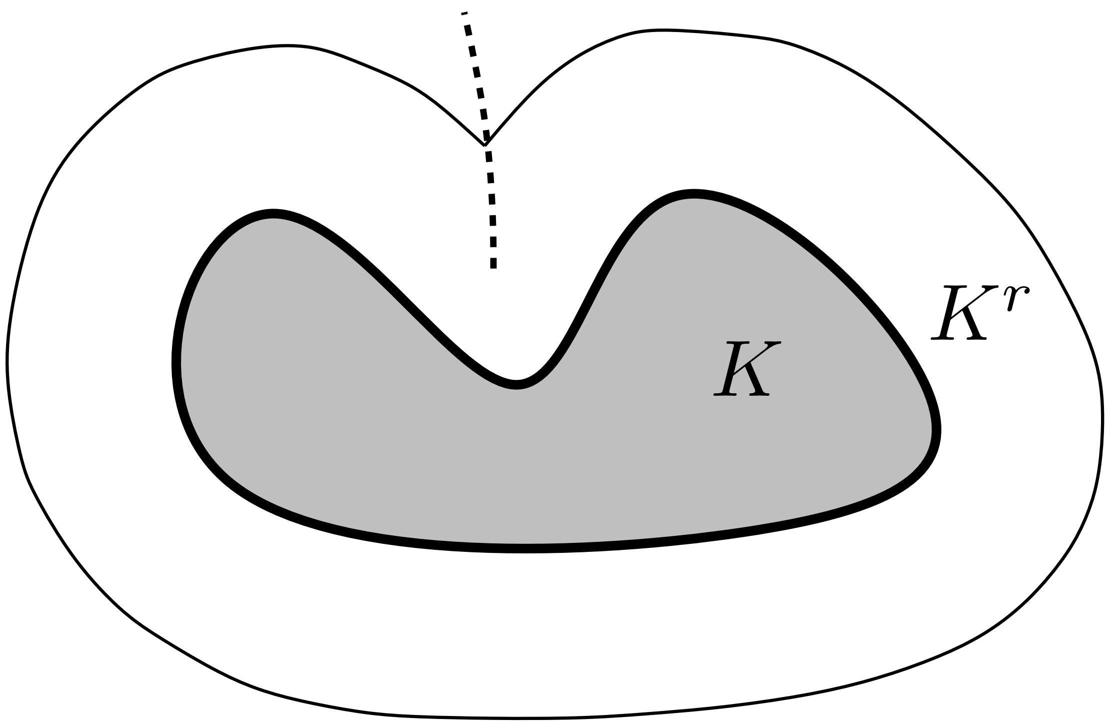

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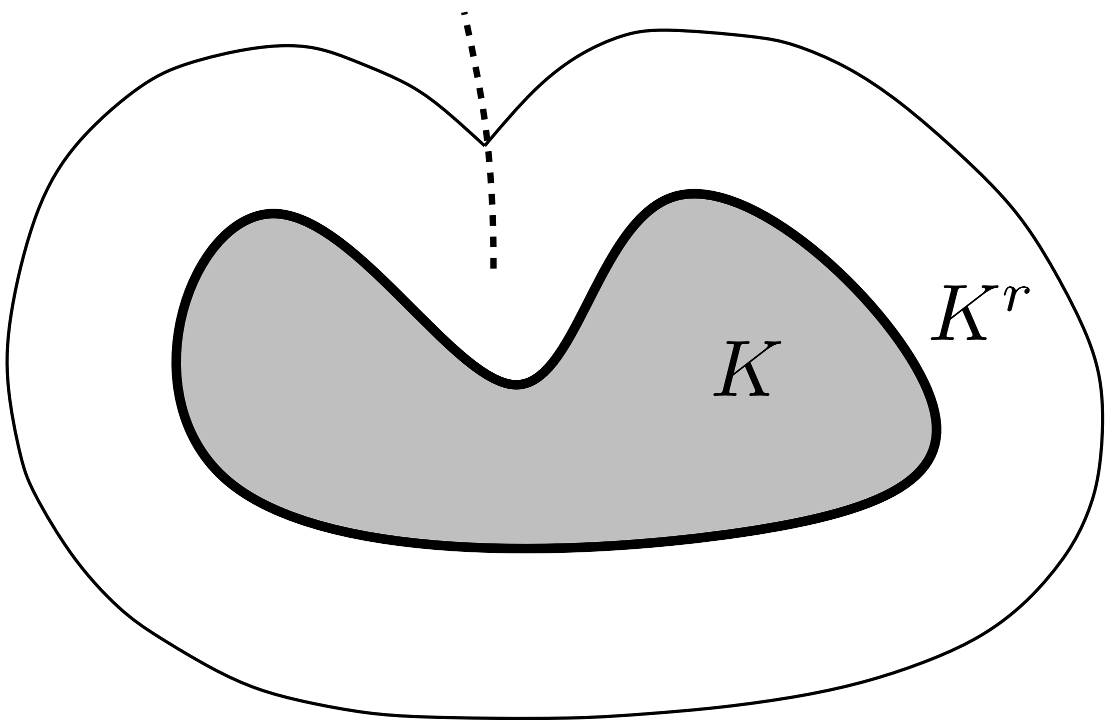

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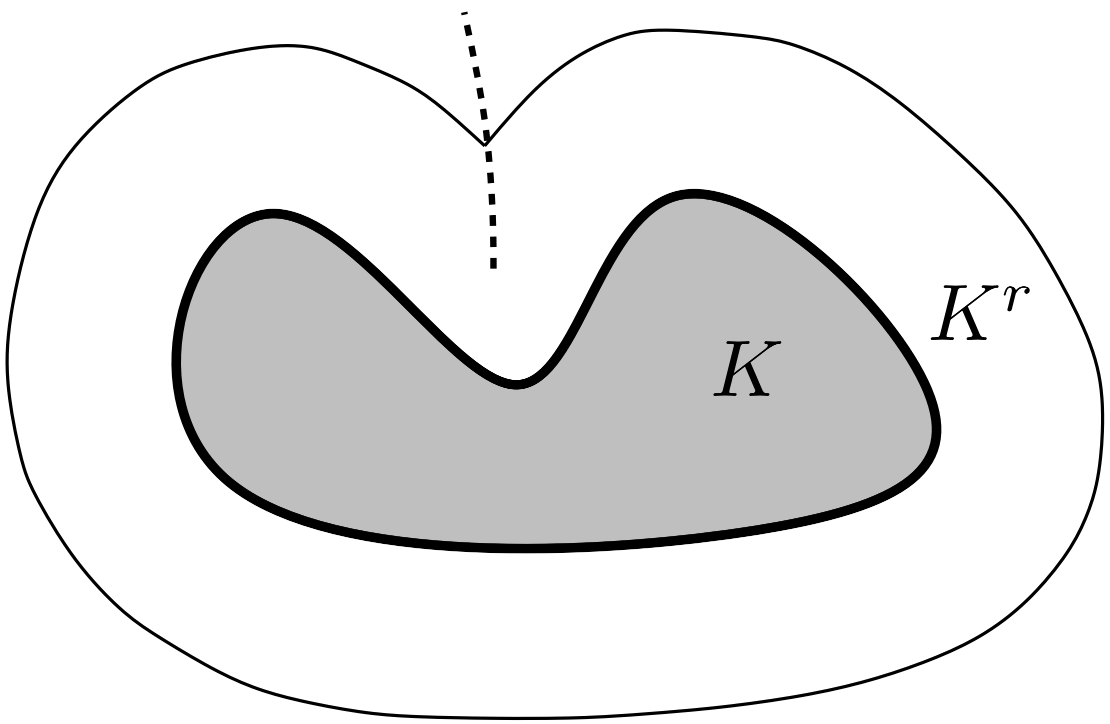

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(ii) If M is smooth, min. curvature radius of $M \geq \operatorname{reach}(M) > 0$.

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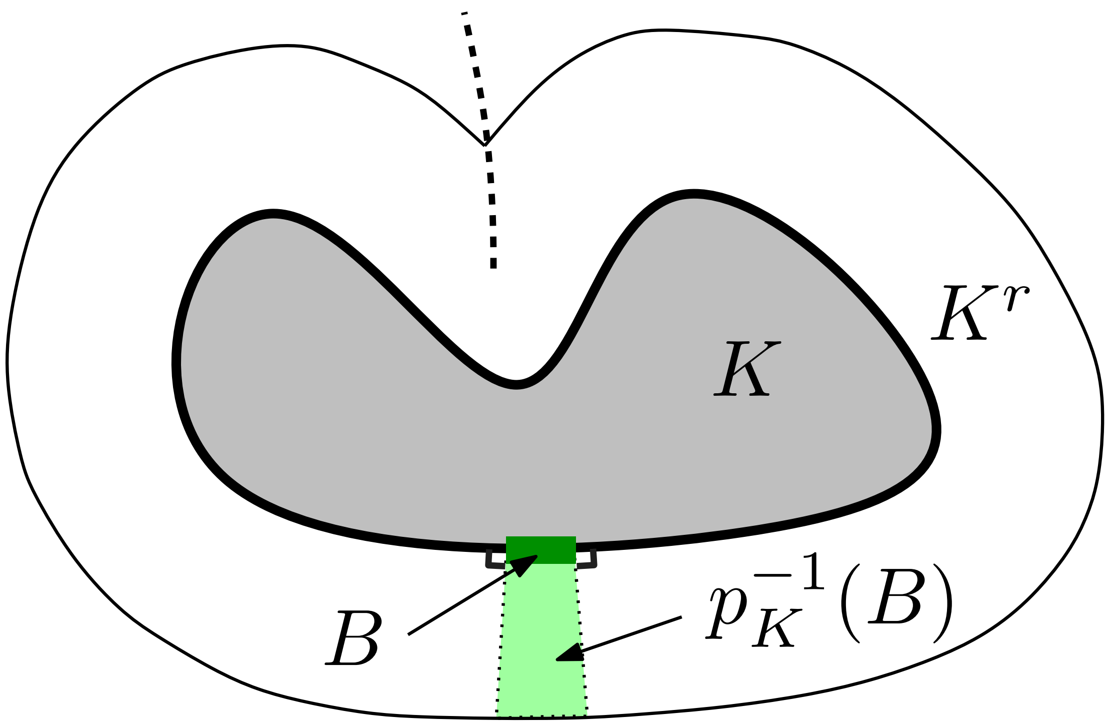

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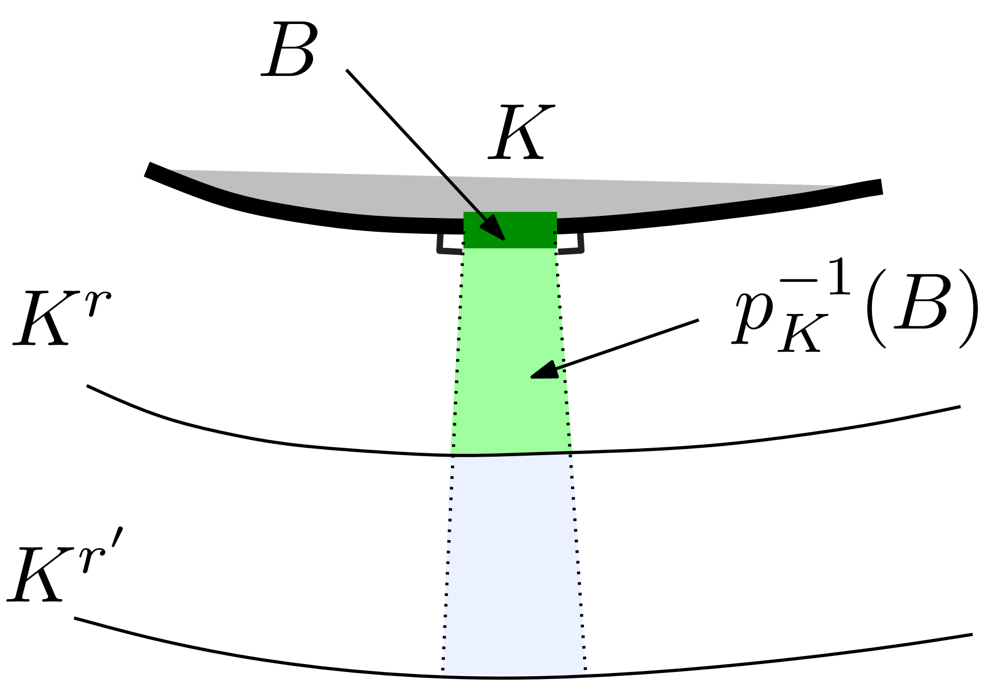

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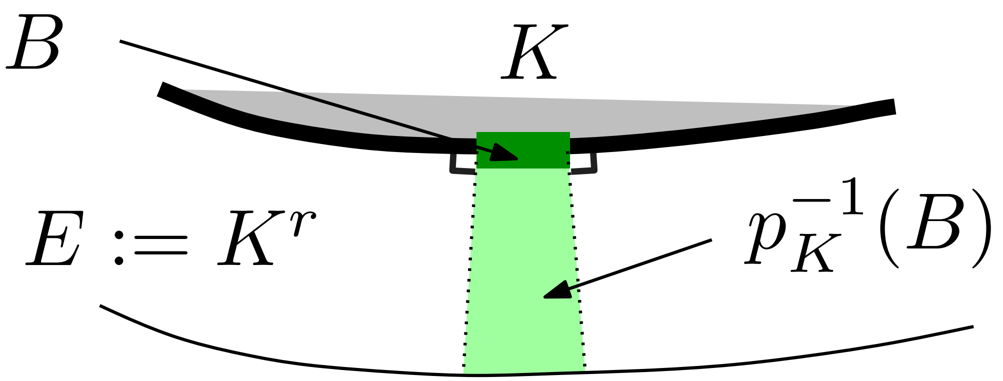
Federer's tube formula: Suppose $R := \mathrm{reach}(K) > 0$. For all subset B of K, the map

$$r \mapsto \mathcal{H}^{\mathrm{d}}(K^r \cap \mathrm{p}_K^{-1}(B))$$

is a polynomial of degree d on $[0, \mathrm{reach}(K)]$.

Boundary measures



Definition: The *boundary measure* of K wrt a domain E is defined for $B \subseteq K$ by

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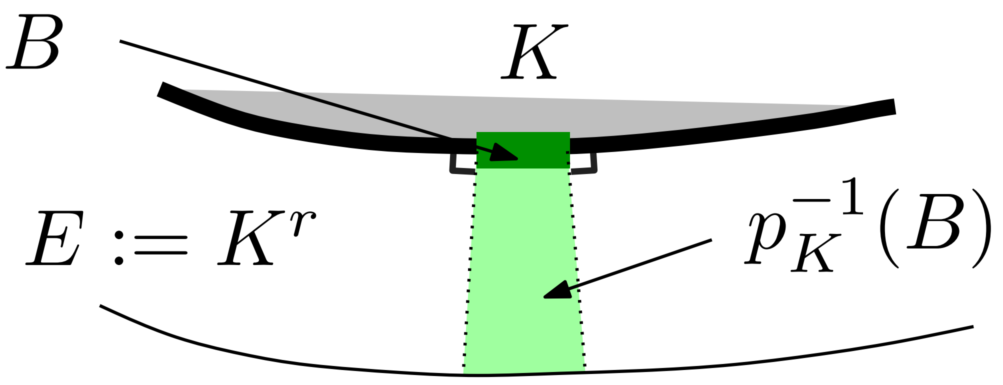

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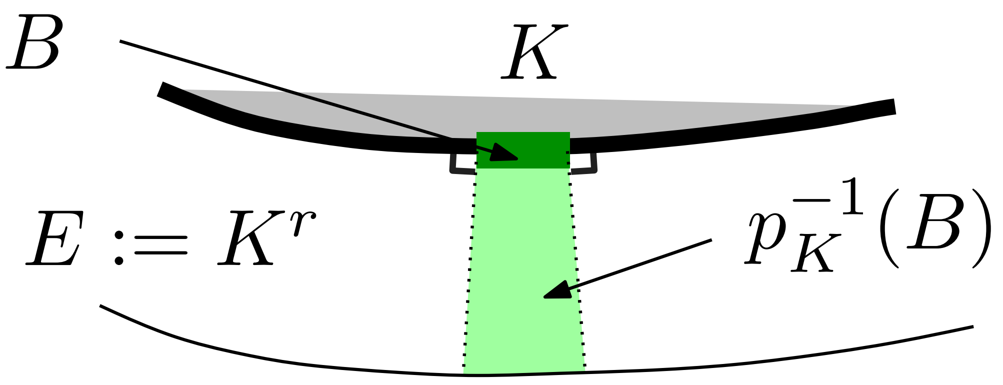

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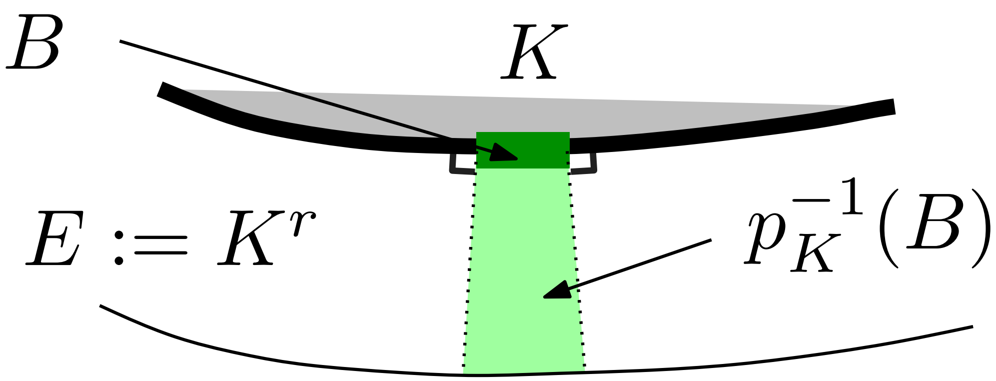

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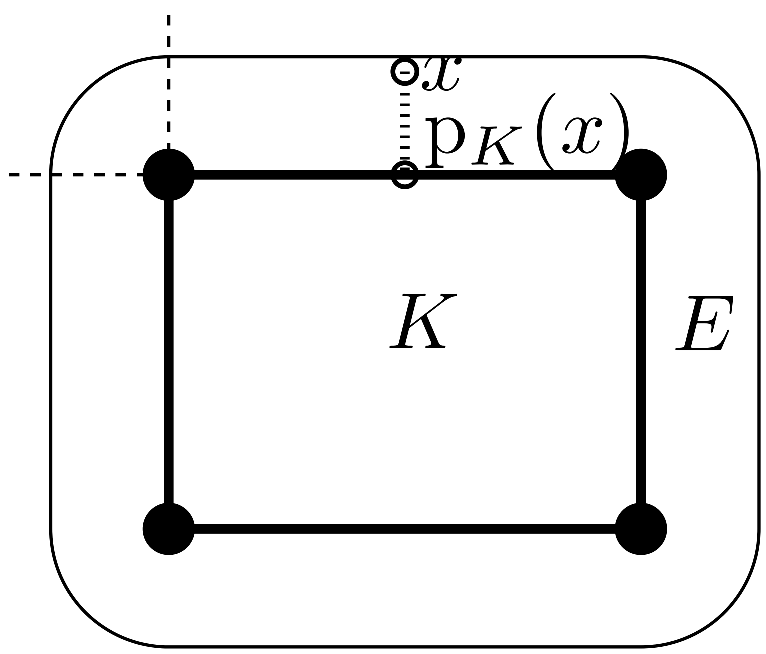
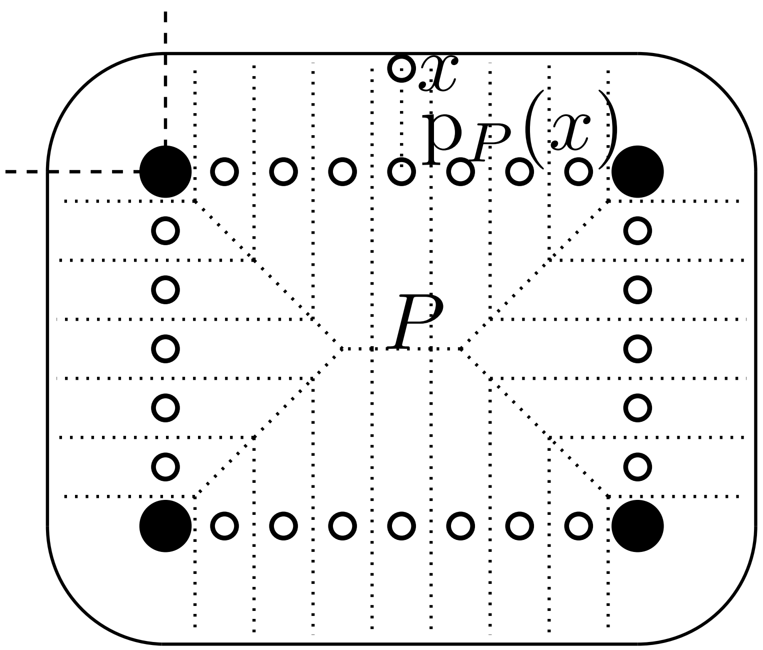
Example:





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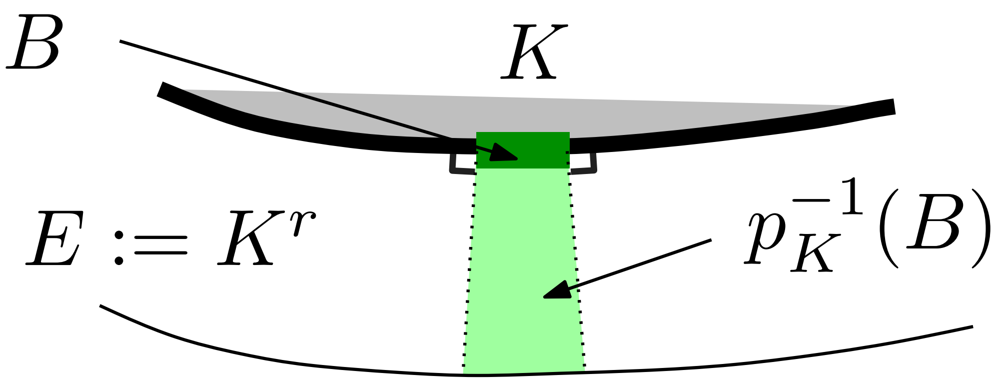

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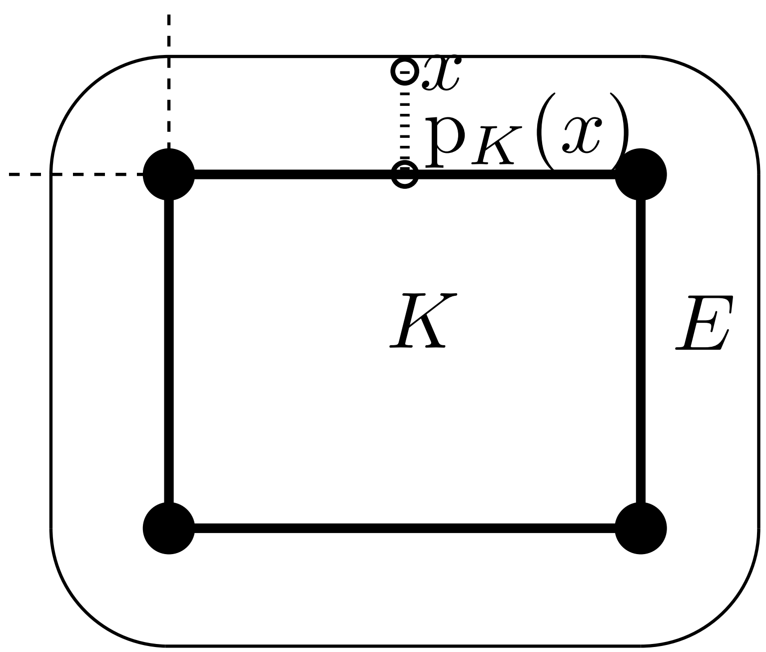

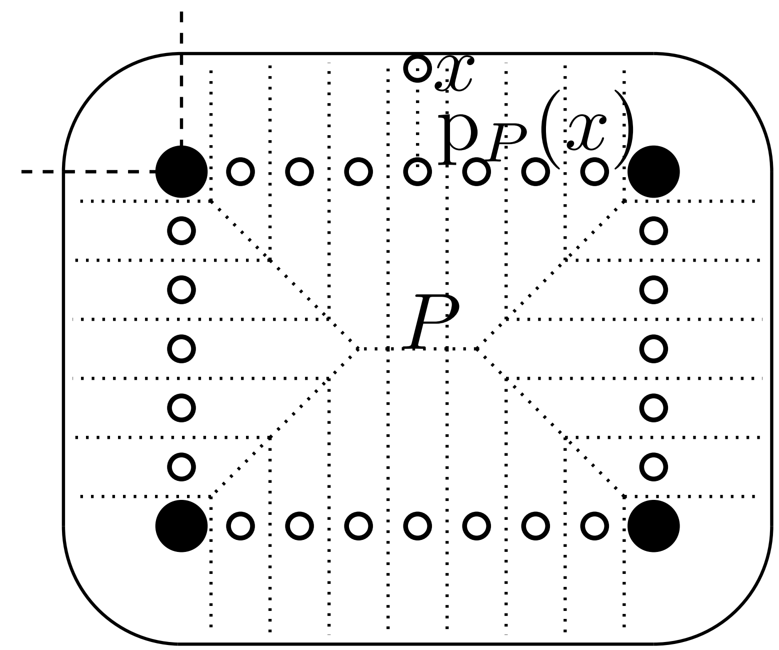

Question: What is the dependence of $\mu_{K,E}$ on K ? For what distance ?

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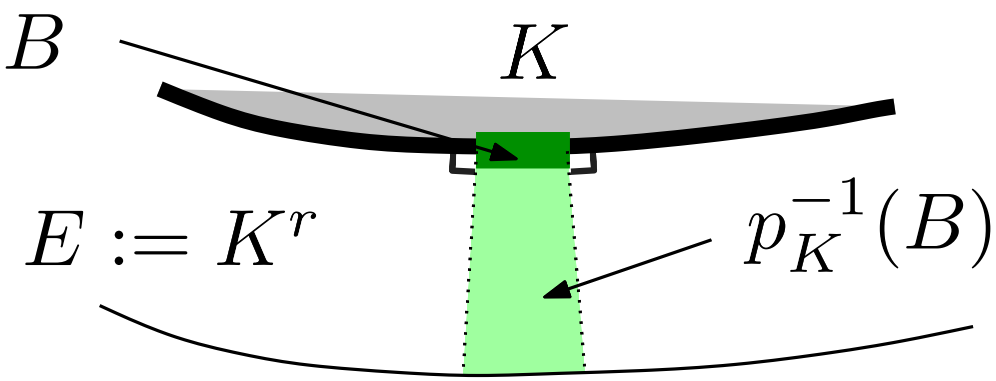

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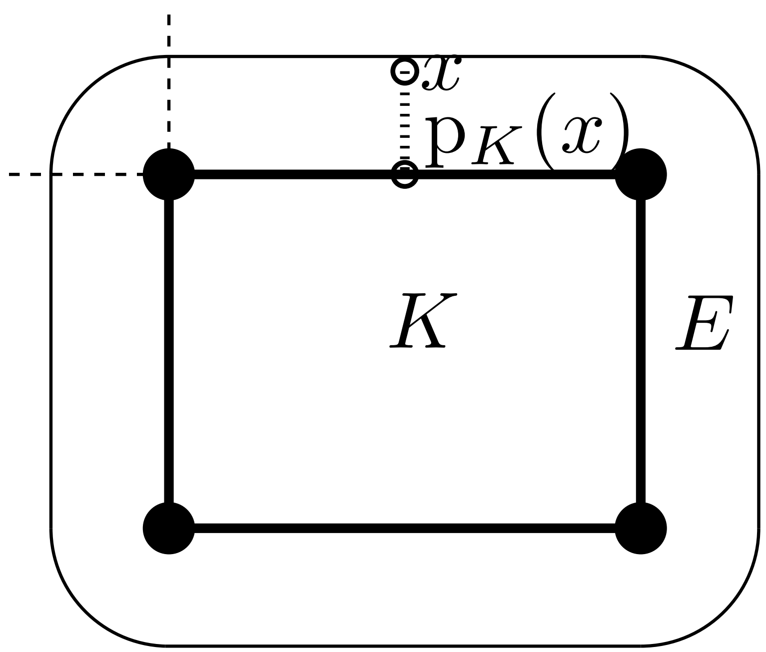

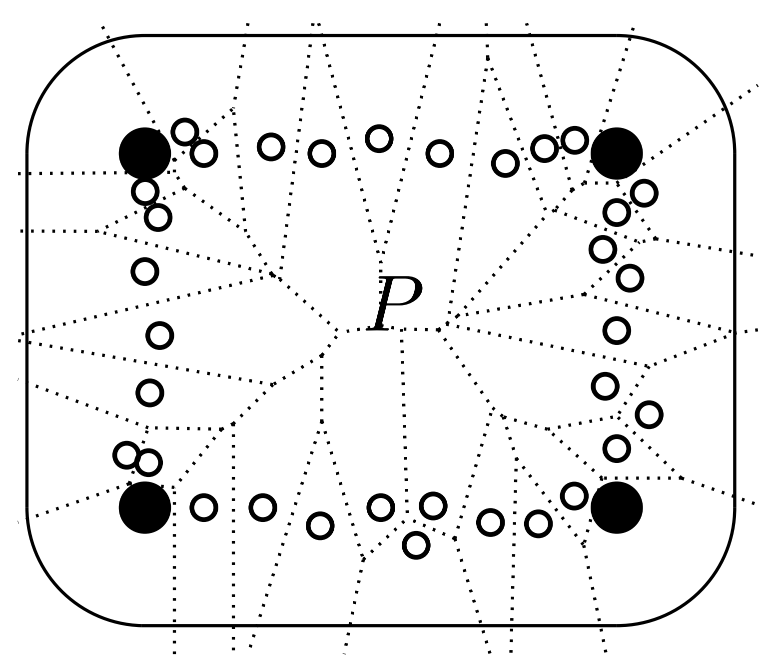

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$$\mathrm{BL}_1 := \{\chi : \mathbb{R}^d \to \mathbb{R};\ \text{1-Lipschitz}, \|\chi\|_\infty \leq 1\}$$

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Bounded-Lipschitz distance: For $\mu, \nu =$ measures with finite mass,

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- If $X \subseteq \mathrm{B}(0, r)$ with $r \geq 1$, and μ, ν are probability measures on X,

$$\mathrm{W}_1(\mu, \nu)/r \le \mathrm{d_{bL}}(\mu, \nu) \le \mathrm{W}_1(\mu, \nu)$$

where W_1 is the Wasserstein distance.

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- **Lemma:** $\mathrm{d}_{\mathrm{bL}}(\mu_{K,E}, \mu_{L,E}) \leq \|\mathrm{p}_K - \mathrm{p}_L\|_{\mathrm{L}^1(E)}$.

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change of variable formula

$$\sup_{\chi \in \mathrm{BL}_1} \left| \int_K \chi(p) \,\mathrm{d}\,\mu_{K,E}(p) - \int_K \chi(p) \,\mathrm{d}\,\mu_{L,E}(p) \right|$$

$$= \sup_{\chi \in \mathrm{BL}_1} \left| \int_E \chi(\mathrm{p}_K(x)) - \chi(\mathrm{p}_L(x)) \,\mathrm{d}\,\mathcal{H}^d(x) \right|$$

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$$= \sup_{\chi \in \mathrm{BL}_1} |\int_E \chi(\mathrm{p}_K(x)) - \chi(\mathrm{p}_L(x)) \,\mathrm{d}\,\mathcal{H}^d(x)|$$

$$\leq \|\mathrm{p}_K - \mathrm{p}_L\|_{\mathrm{L}^1(E)}$$

χ is 1-Lipschitz

Nonquantitative stability of curvature measures

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Goal: Show $\|p_K - p_L\|_{\mathrm{L}^1(E)} = \mathrm{O}(\mathrm{d_H}(K, L)^\alpha)$ for arbitrary compact sets

Optimal stability for boundary measures 1/3

Theorem: Let $K, L \subseteq \mathbb{R}^d$ be compact sets and E a bounded domain

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[Chazal, Cohen-Steiner, M. 2007]

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→ defined through polynomial fitting, i.e.

$\mu_{L,L^{r_\ell}} = \sum_{i=0}^d \tilde{\Phi}_L^{d-i} r_\ell^i$ for fixed $0 < r_0 < \ldots < r_d < R$

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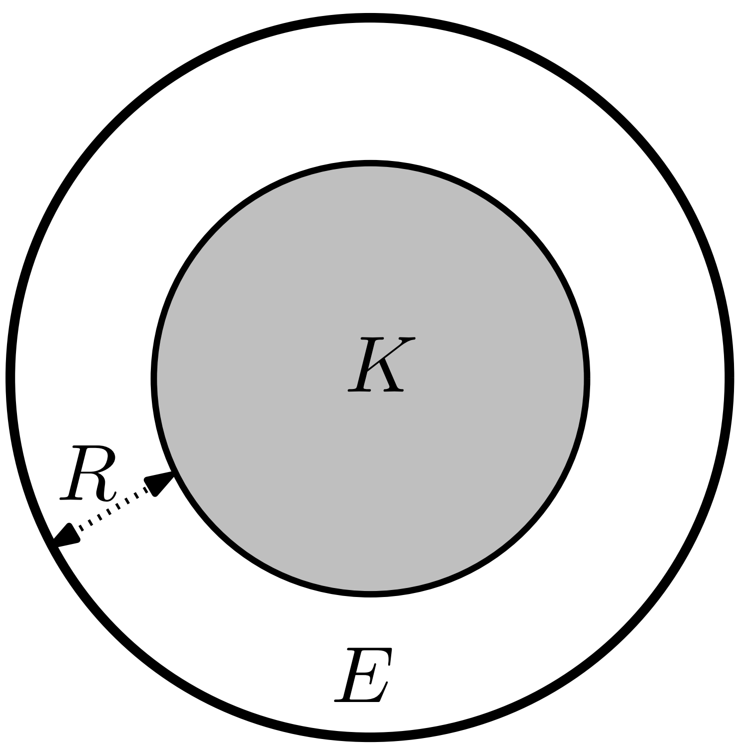

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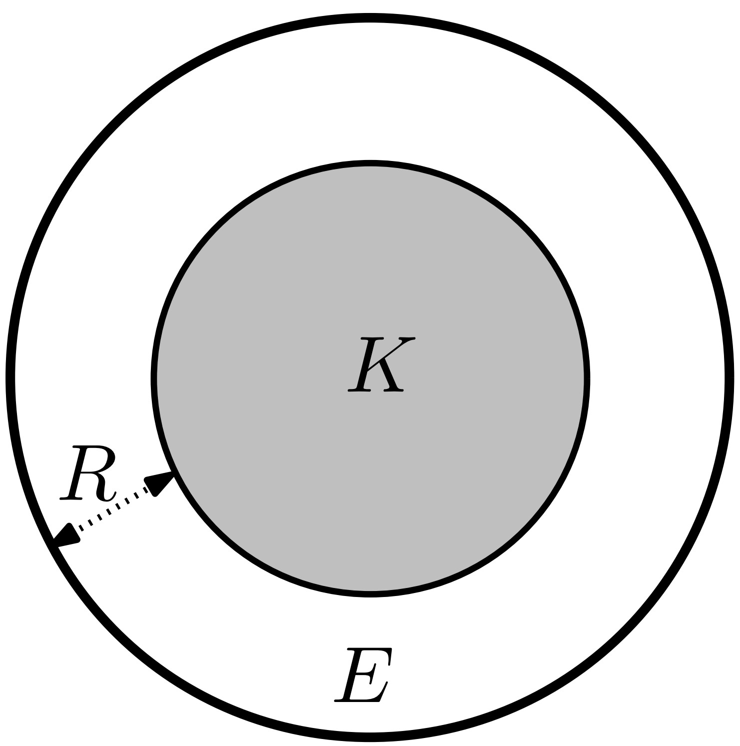

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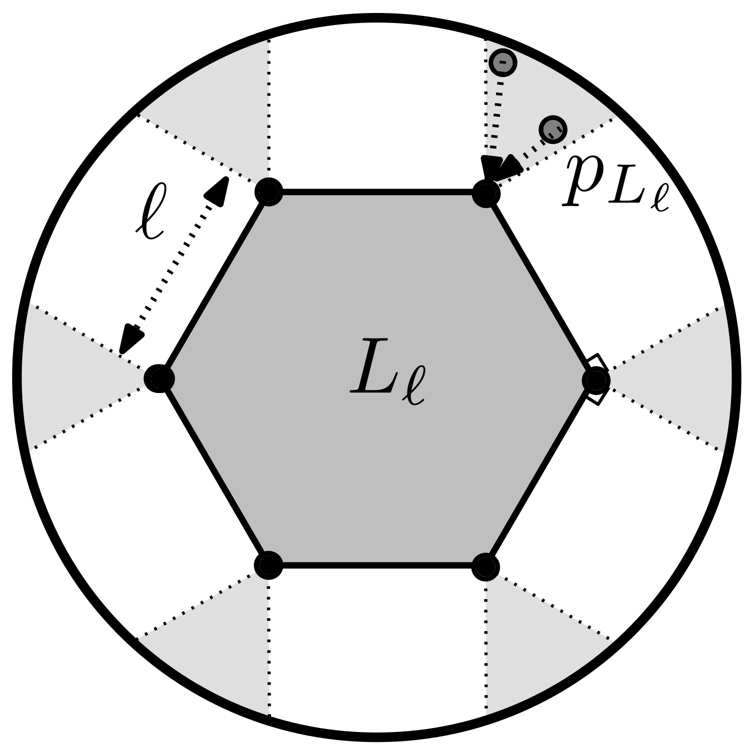

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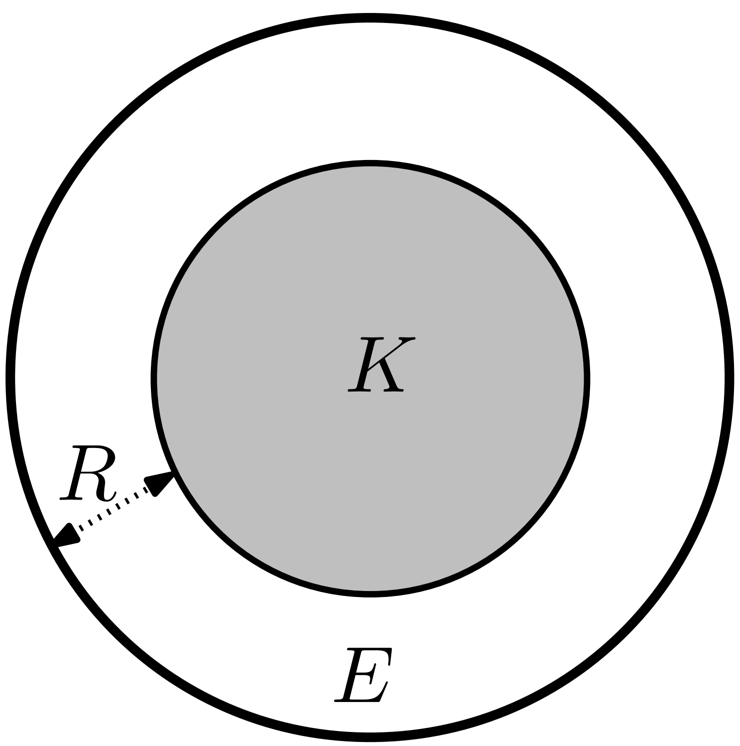

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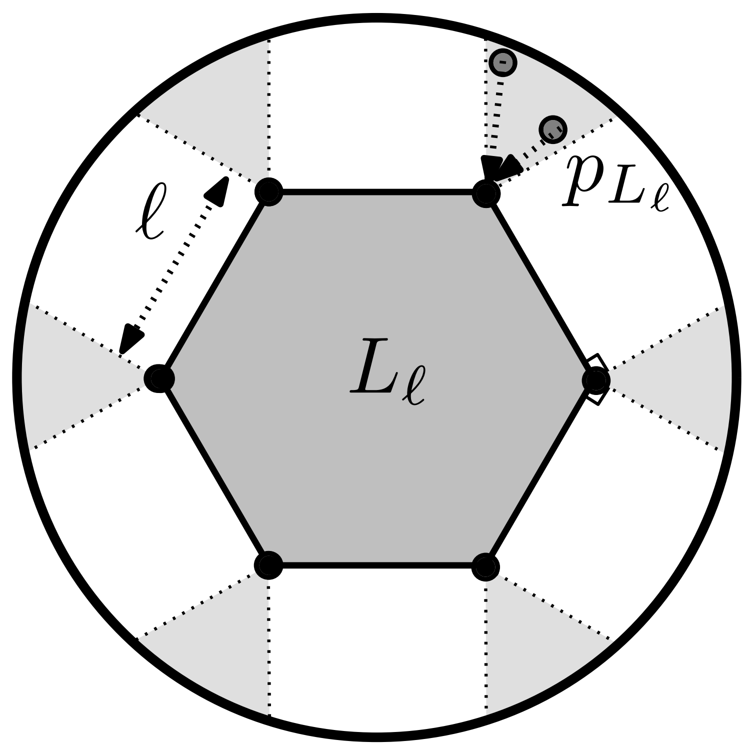

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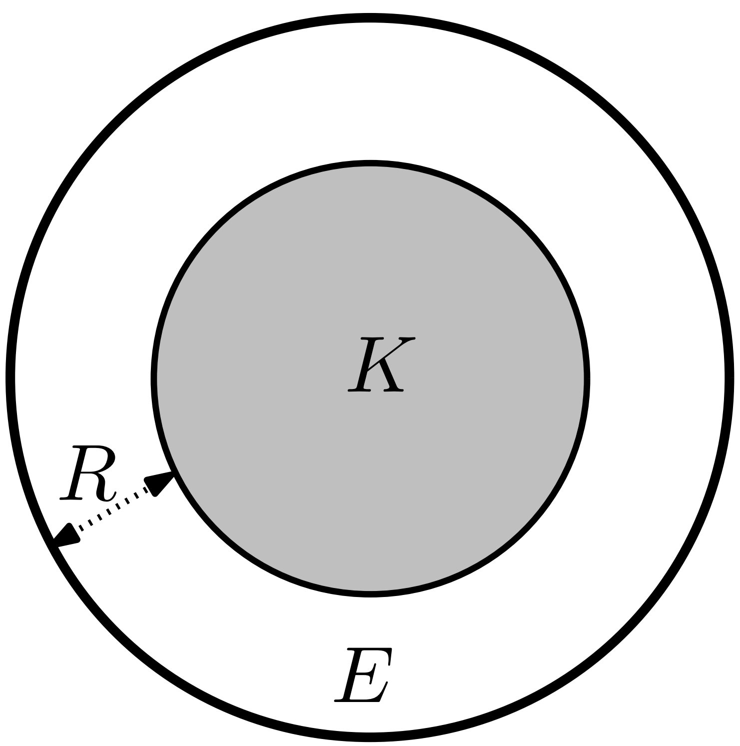

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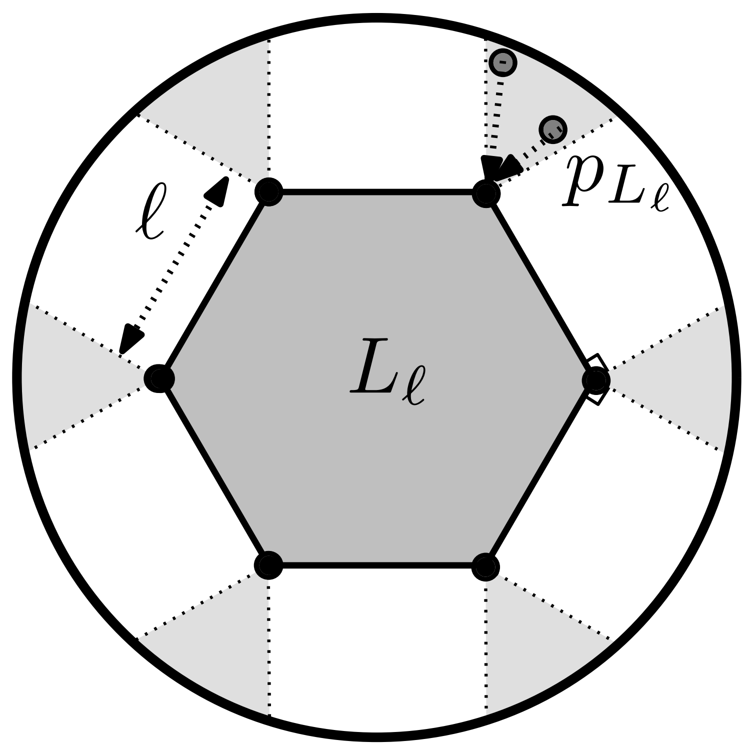

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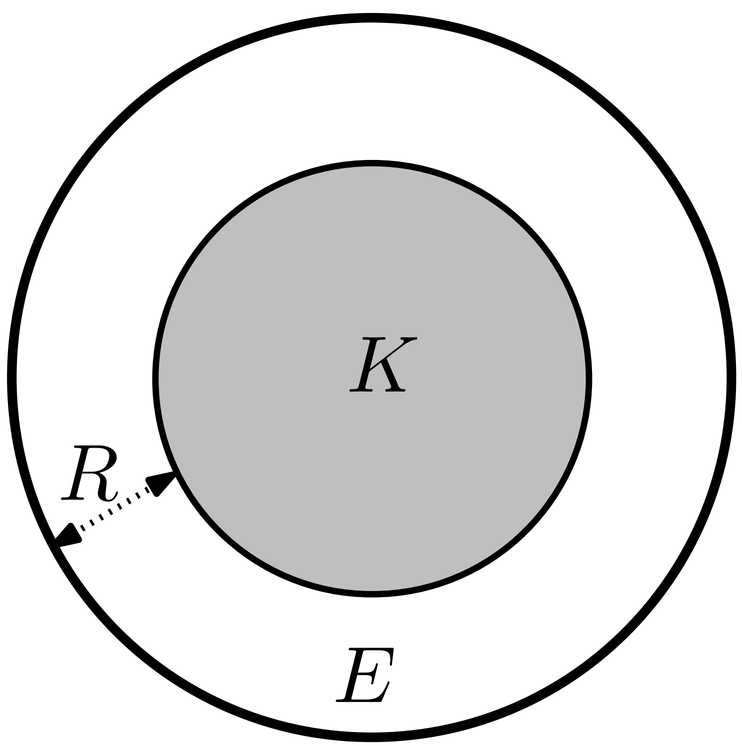

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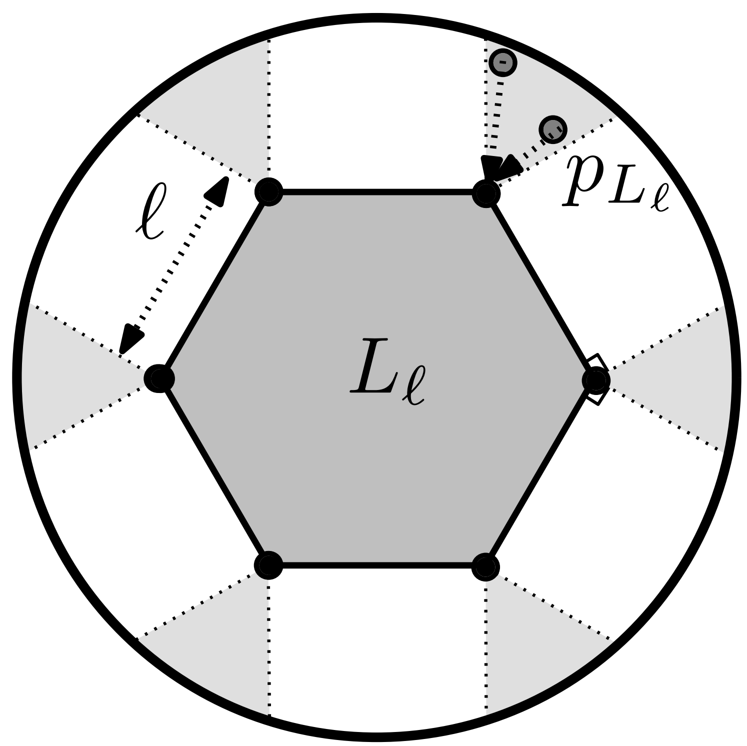

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$\longrightarrow$ **1-semiconcavity** of the distance function to a compact set.

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Step 3: With $u = v_K$, $v = v_L$, $\|\nabla v_K\|_{\mathrm{L}^\infty(E)} + \|\nabla v_K\|_{\mathrm{L}^\infty(E)} = c_K$

$$\|\mathrm{p}_K - \mathrm{p}_L\|^2_{\mathrm{L}^2(E)} = \|\nabla v_K - \nabla v_L\|^2_{\mathrm{L}^2(E)} \leq c_{E,K}\boxed{\|v_K - v_L\|_{\mathrm{L}^\infty(E)}}$$

$$= \tfrac{1}{2}\|\,\mathrm{d}_K^2 - \mathrm{d}_L^2\,\|_{\mathrm{L}^\infty(E)} \leq \tfrac{1}{2}\|\,\mathrm{d}_K - \mathrm{d}_L\,\|_{\mathrm{L}^\infty(E)} \cdot \|\,\mathrm{d}_K + \mathrm{d}_L\,\|_{\mathrm{L}^\infty(E)}$$

$$\leq c_K\, \mathrm{d_H}(K,L)$$

Optimal stability for boundary measures 3/3

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Stokes

$$\int_E \|\nabla u - \nabla v\|^2 \overset{\text{Stokes}}{=} \int_{\partial E}(u - v)\langle \nabla u - \nabla v | \mathbf{n}_E \rangle - \int_E (u - v)\Delta(u - v)$$

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finally: $\int_E |\Delta u| = \int_E \Delta u$

convexity

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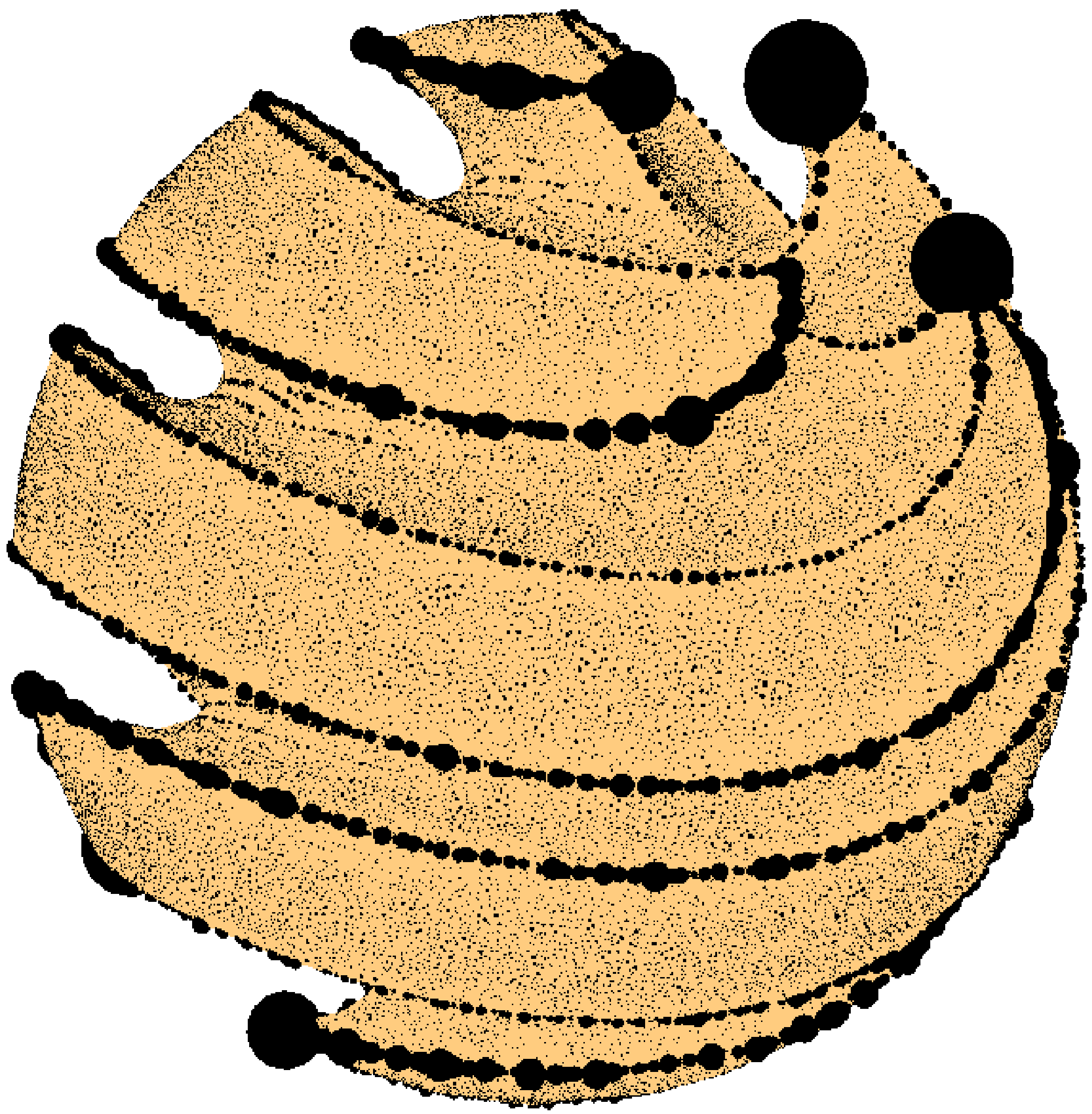
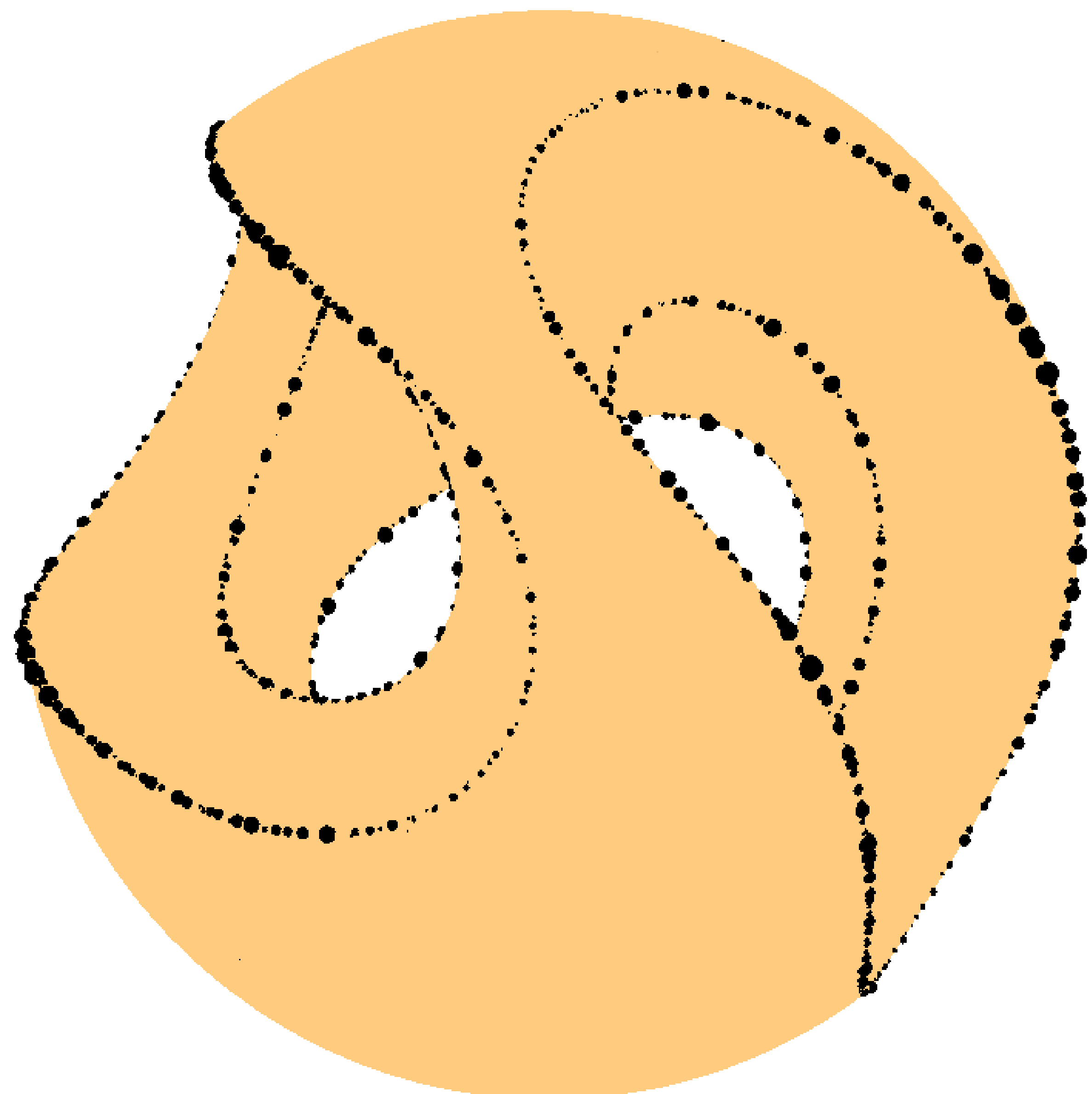
Example of boundary measures

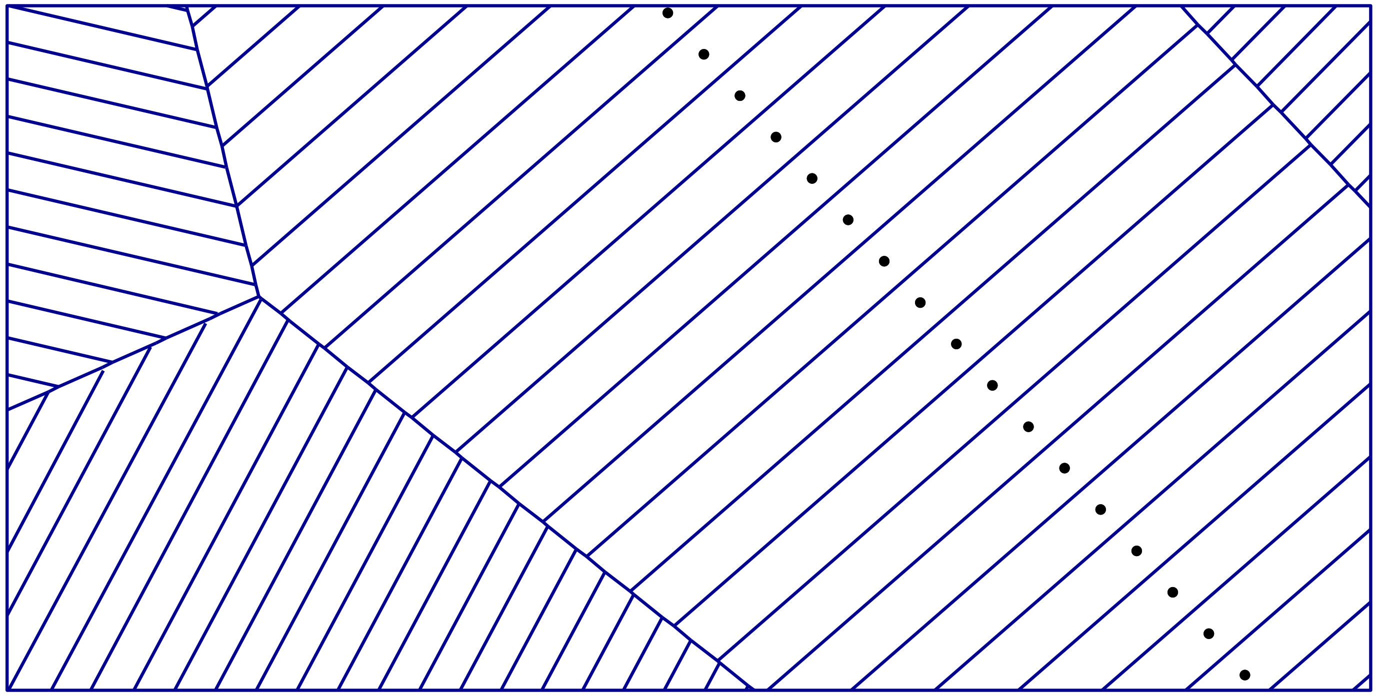




[Chazal-Cohen-Steiner-M. '07]

2. Voronoi covariance measure

Voronoi-based normal estimation

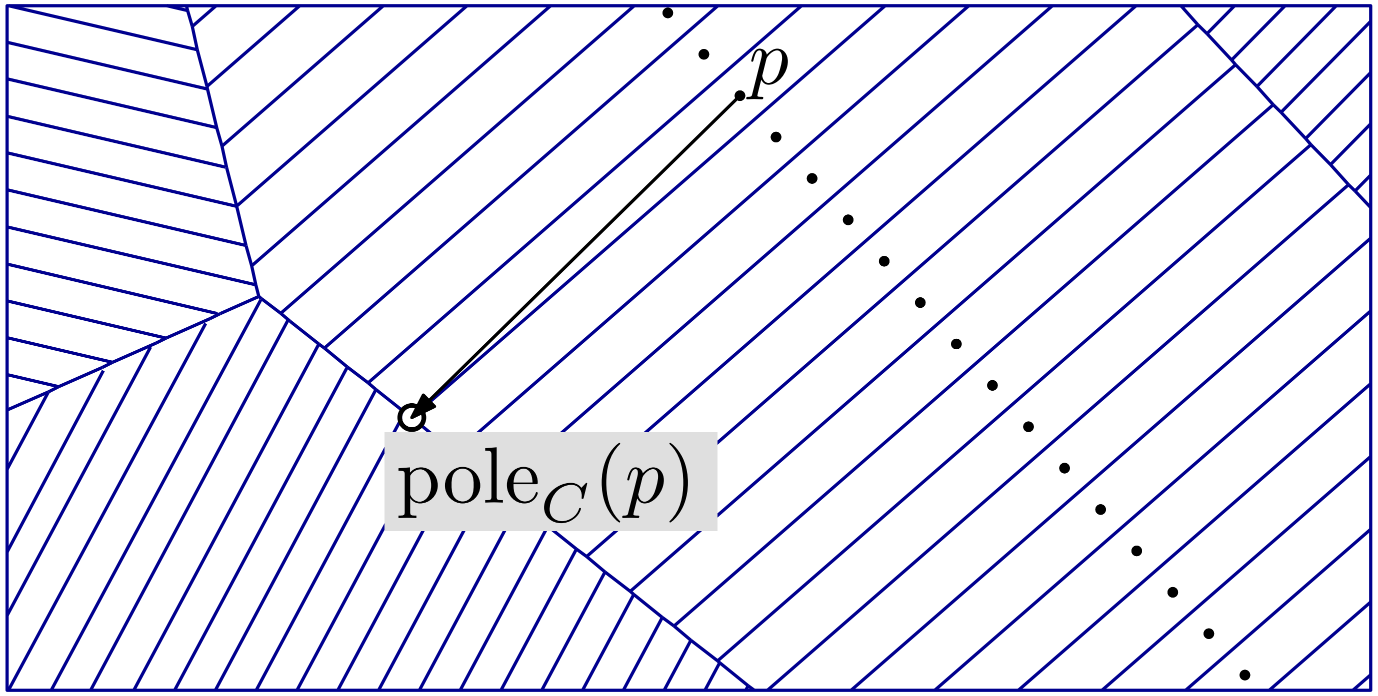


$$P = \{p_1, \ldots, p_N\} \subseteq \mathbb{R}^d$$

Voronoi cell:

$$\mathrm{Vor}_P(p) = \{x \in \mathbb{R}^d ; \forall q \in P, \|x - p\| \leq \|x - q\|\}$$

Voronoi-based normal estimation



$\mathrm{pole}_C(p) :=$ farthest point to p in $\mathrm{Vor}_C(p)$

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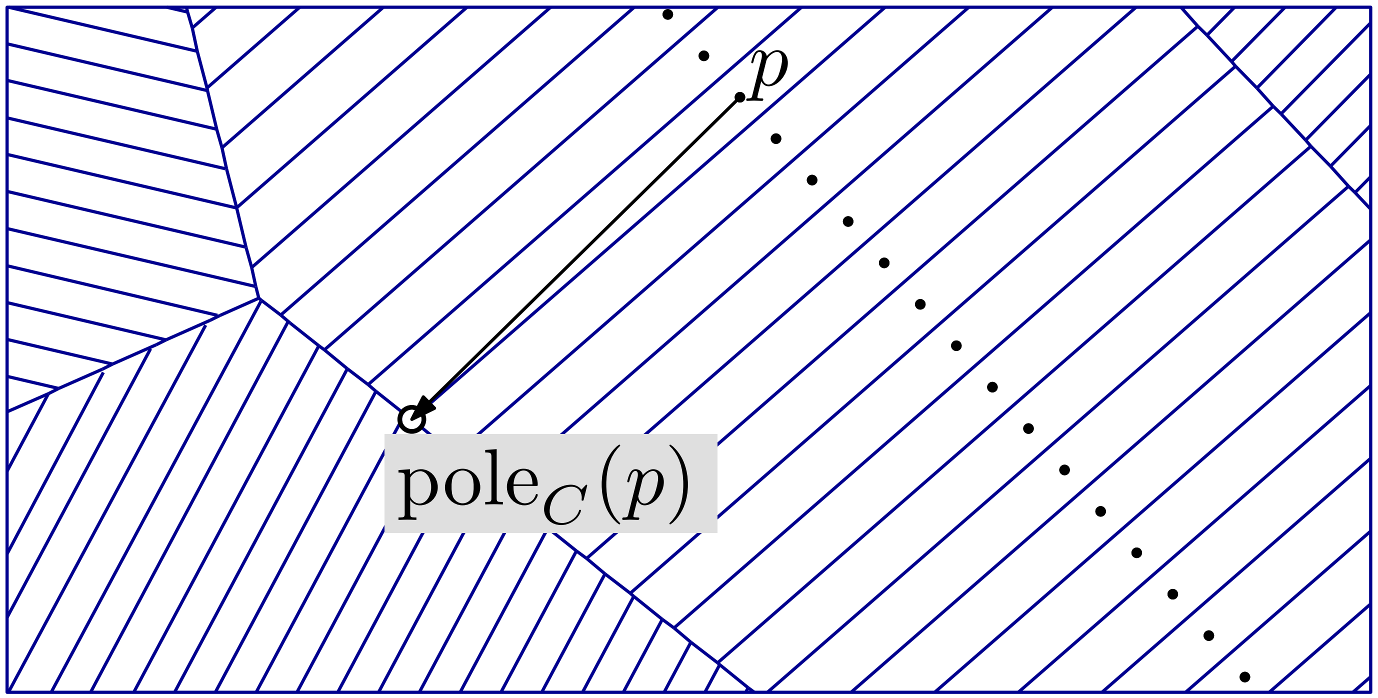

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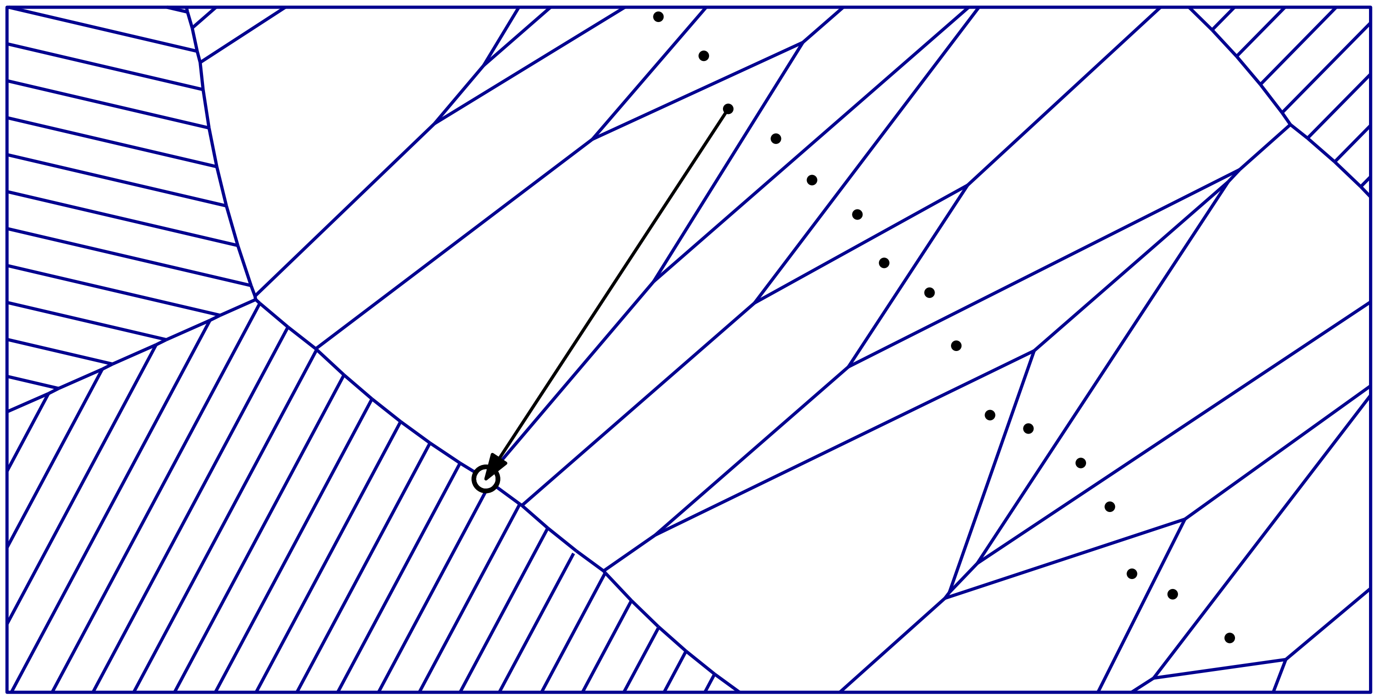
If C is a **noiseless** ε-sampling of a surface S, the angle between $\mathrm{pole}_C(p) - p$ and the normal of S at p is $\mathrm{O}(\varepsilon)$.

[Amenta, Bern 1999]

Voronoi-based normal estimation



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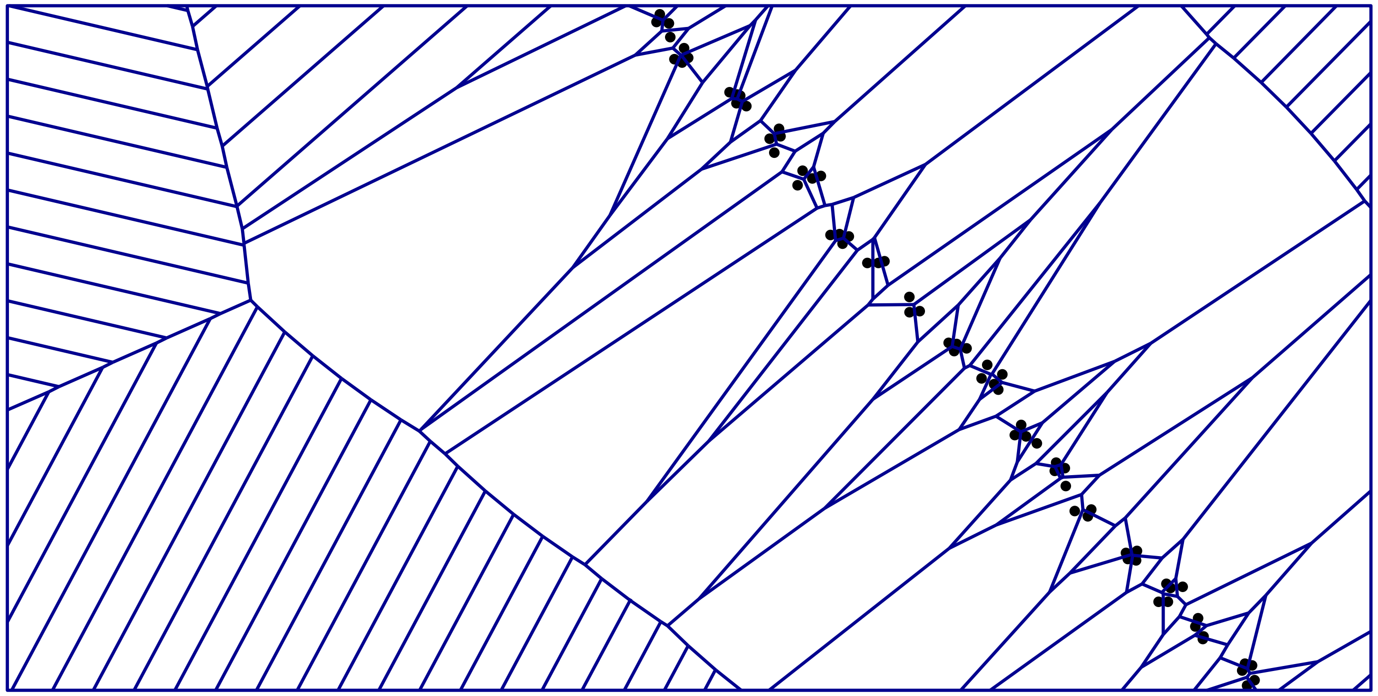

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[Amenta, Bern 1999] [Dey, Sun 2005]

Need of an **integral** quantity to get stability under Hausdorff noise.

Normal estimation based on Voronoi covariance

Covariance matrix: $\operatorname{cov}_p(\Omega) := \int_\Omega (x-p) \otimes (x-p) \, \mathrm{d}\, x.$

$[v \otimes v]_{ij} := v_i v_j$

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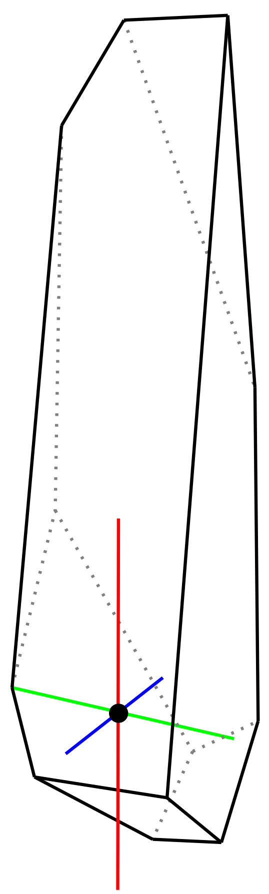
Algorithm:

- Consider the covariance matrix $\mathrm{cov}_{p_i}(\mathrm{Vor}_C(p_i) \cap E)$



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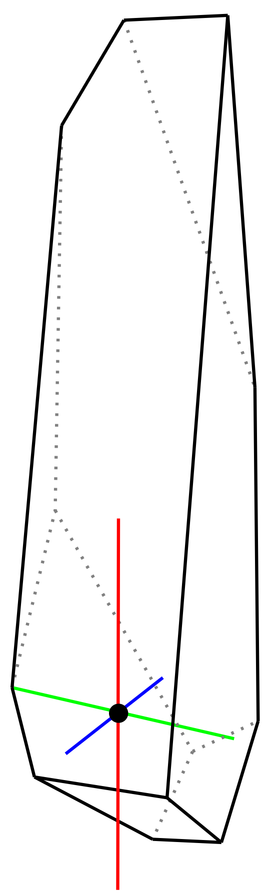
- The normal is estimated by the eigenvector corresponding to the largest eigenvalue (in red).



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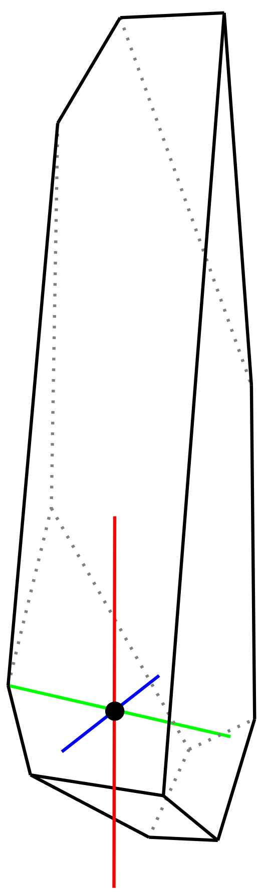
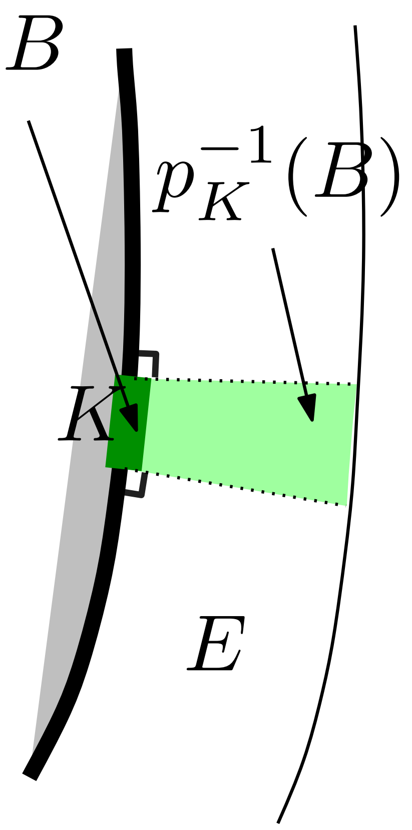


- Consider the covariance matrix $\mathrm{cov}_{p_i}(\mathrm{Vor}_C(p_i) \cap E)$
- The normal is estimated by the eigenvector corresponding to the largest eigenvalue (in red).
- Resilience to noise is achieved by taking union of neighbouring Voronoi cells.

[Alliez, Cohen-Steiner, Tong, Desbruns 2007]

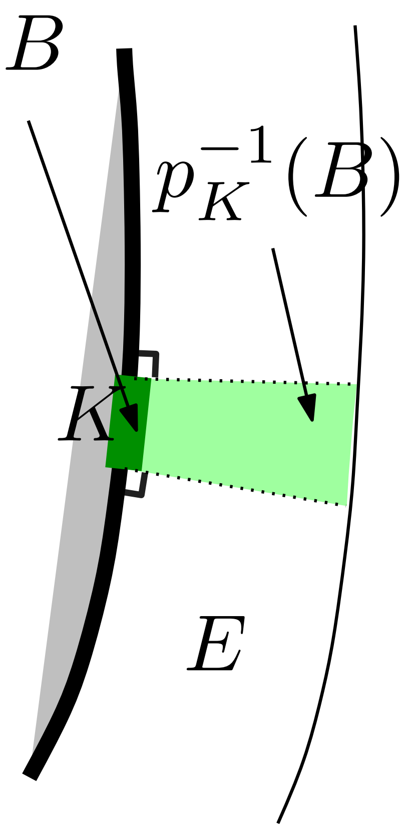
Voronoi covariance measure



The **Voronoi covariance measure** of K wrt a domain E is a tensor-valued measure on $\mathbb{R}^d$. For $B \subseteq \mathbb{R}^d$,

$$\mathcal{V}_{K,E}(B) = \int_{E \cap \mathrm{p}_K^{-1}(B)} (x - \mathrm{p}_K(x)) \otimes (x - \mathrm{p}_K(x)) \, \mathrm{d}\, \mathcal{H}^d(x)$$

Voronoi covariance measure



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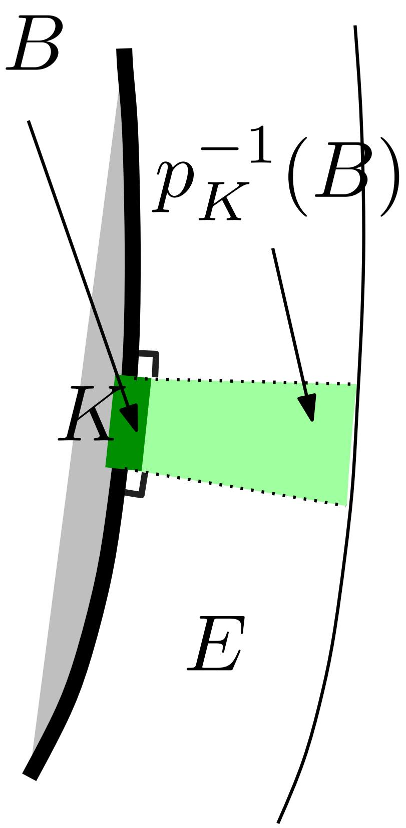
NB: Boundary measure: $\mu_{K,E}(B) = \int_{E \cap \mathrm{p}_K^{-1}(B)} 1 \, \mathrm{d}\, \mathcal{H}^d(x)$

Voronoi covariance measure



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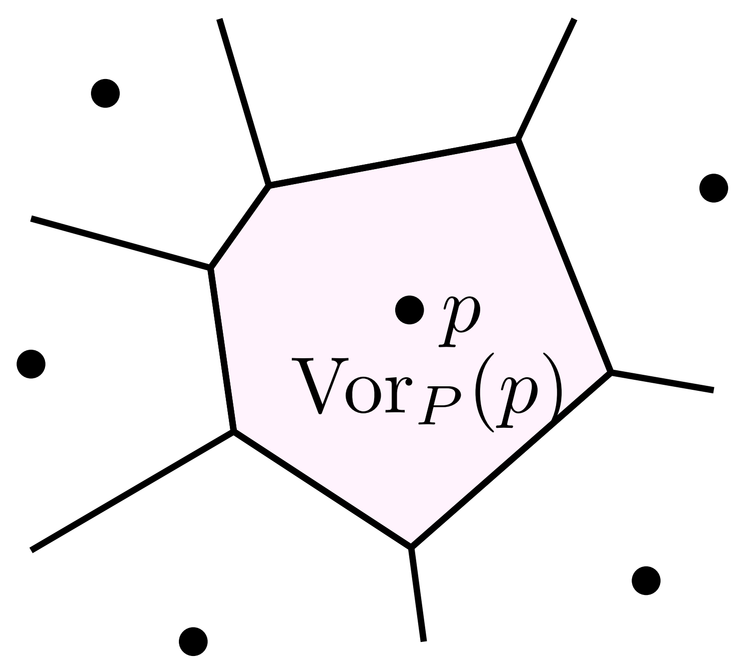
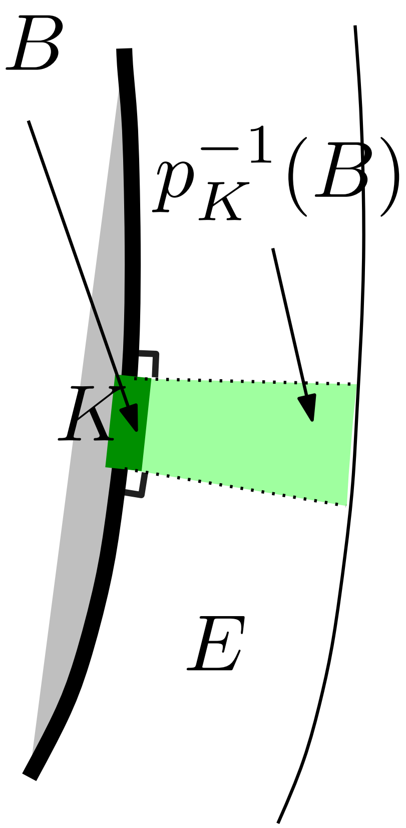
- **Discrete setting:** $P = \{\bullet\} \subseteq \mathbb{R}^d$



Voronoi covariance measure



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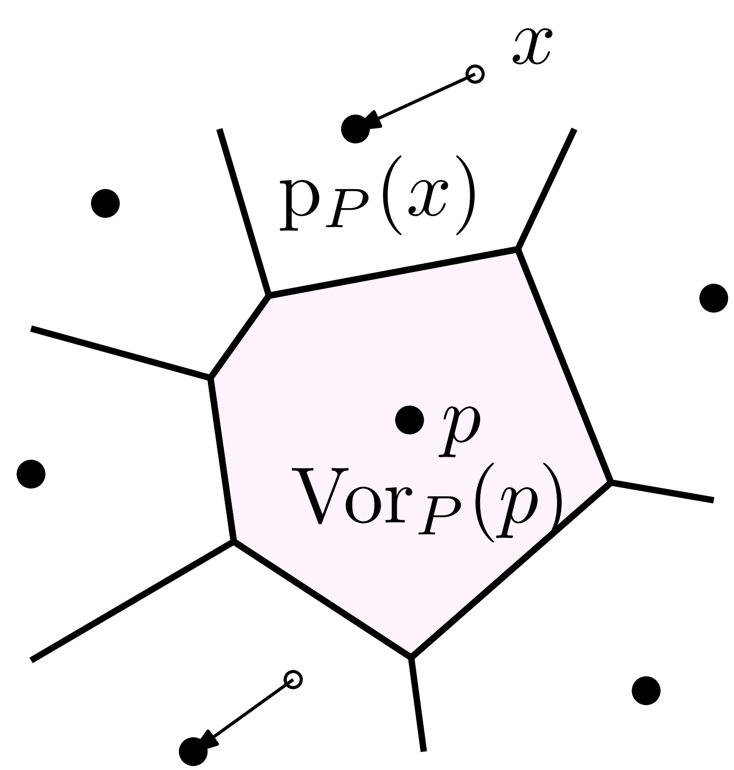
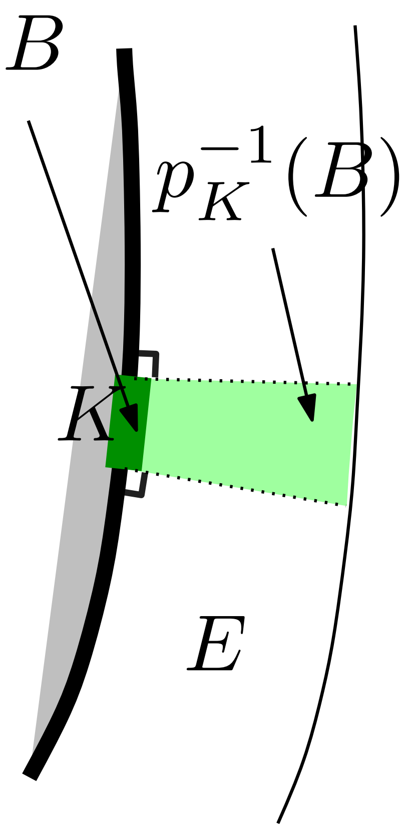


Voronoi covariance measure



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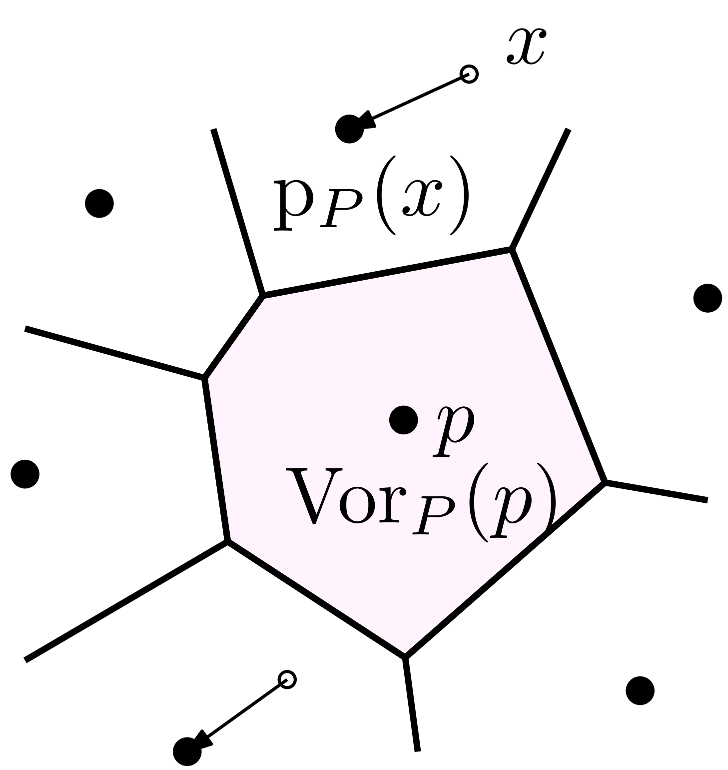
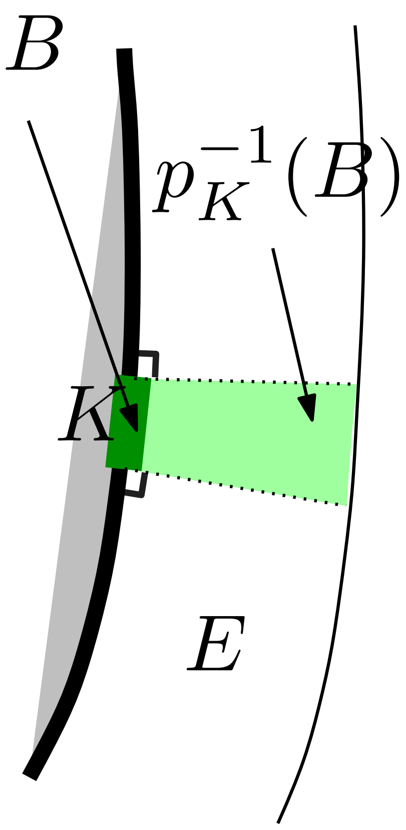


Voronoi covariance measure



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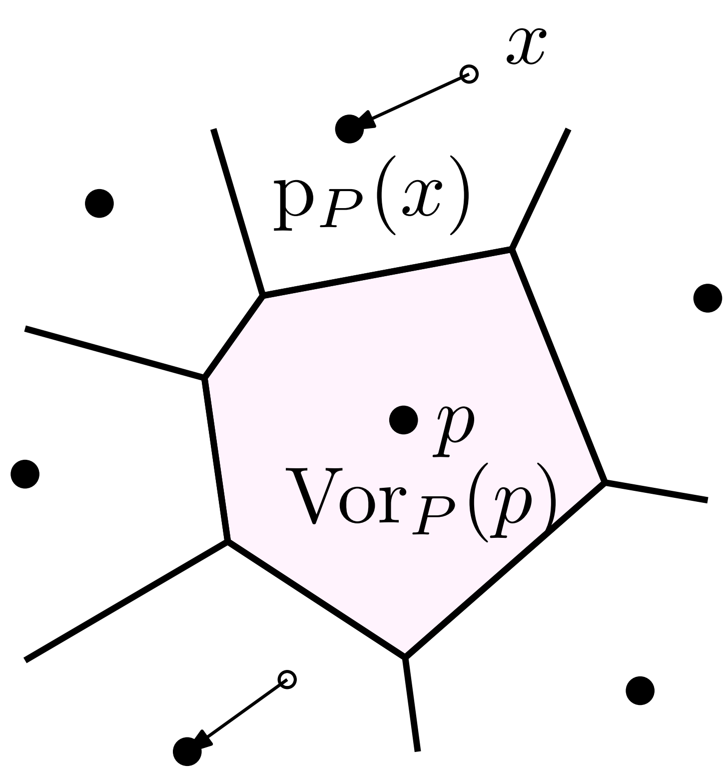

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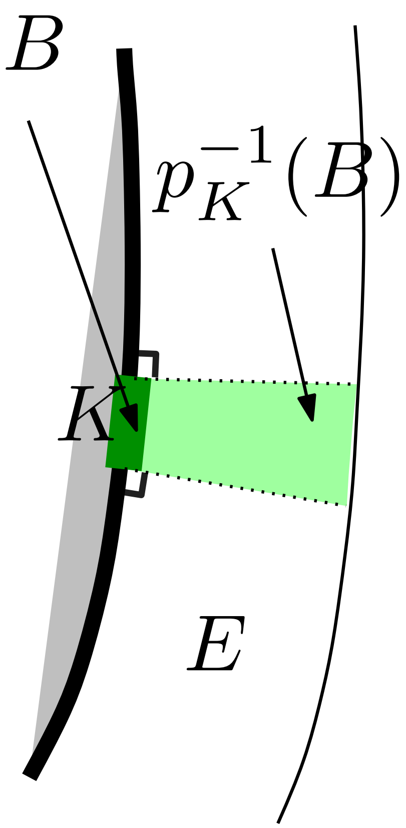
Voronoi covariance measure



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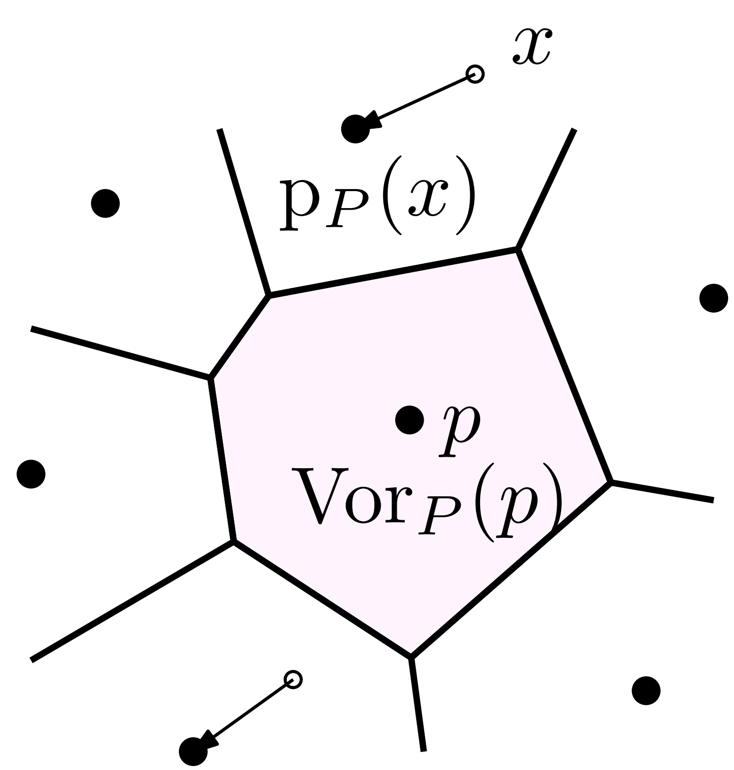

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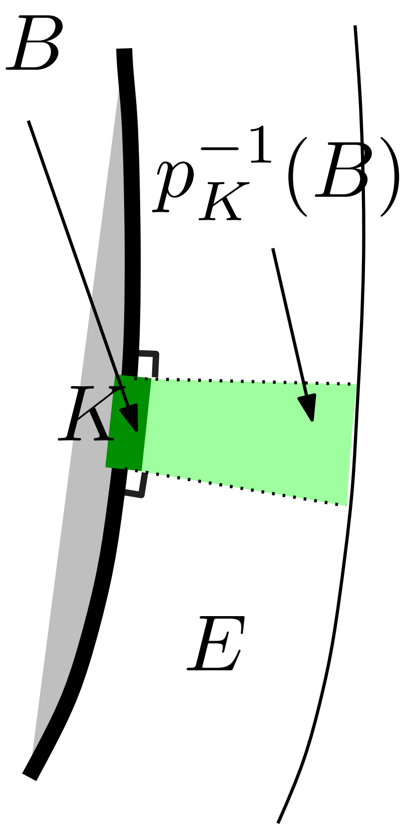
Voronoi covariance measure



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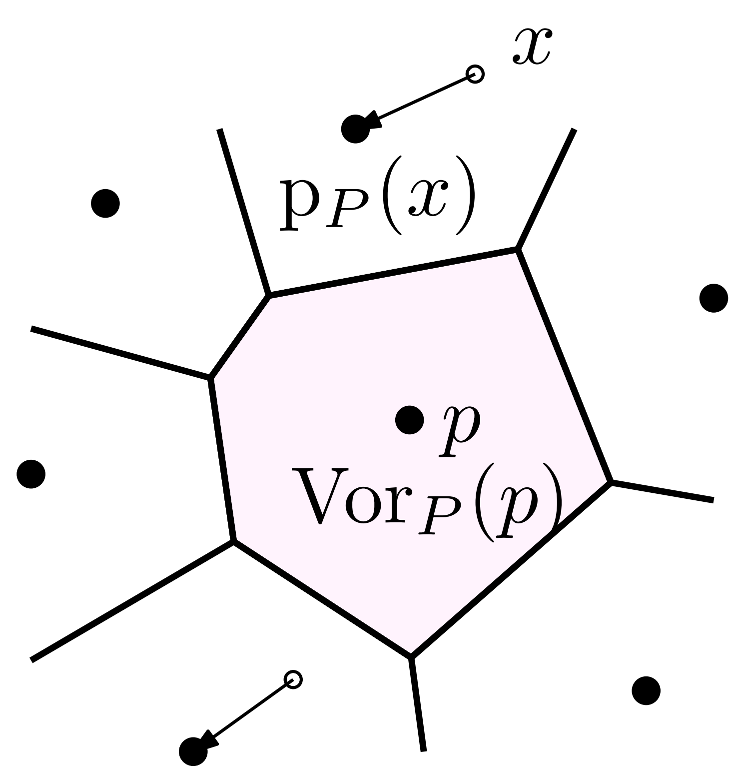

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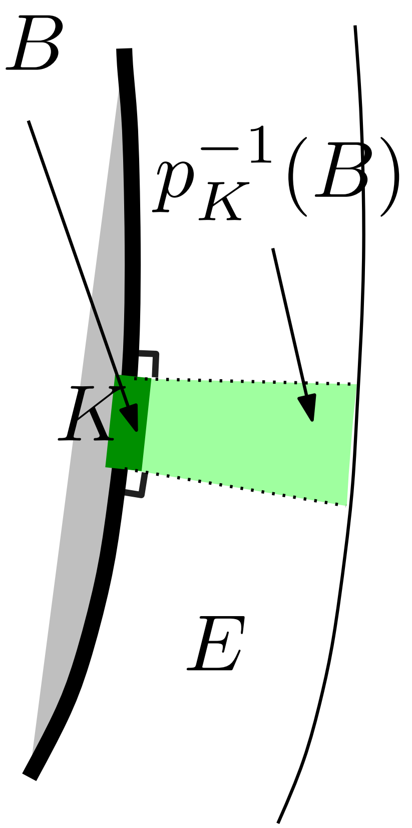
- If $\mathrm{reach}(K) > R$, $\exists \mathcal{V}_K^i$ s.t. $\mathcal{V}_{K,K^r} = \sum_{i=1}^d \mathcal{V}_K^i r^{d-i}$ on $[0, R]$

Voronoi covariance measure



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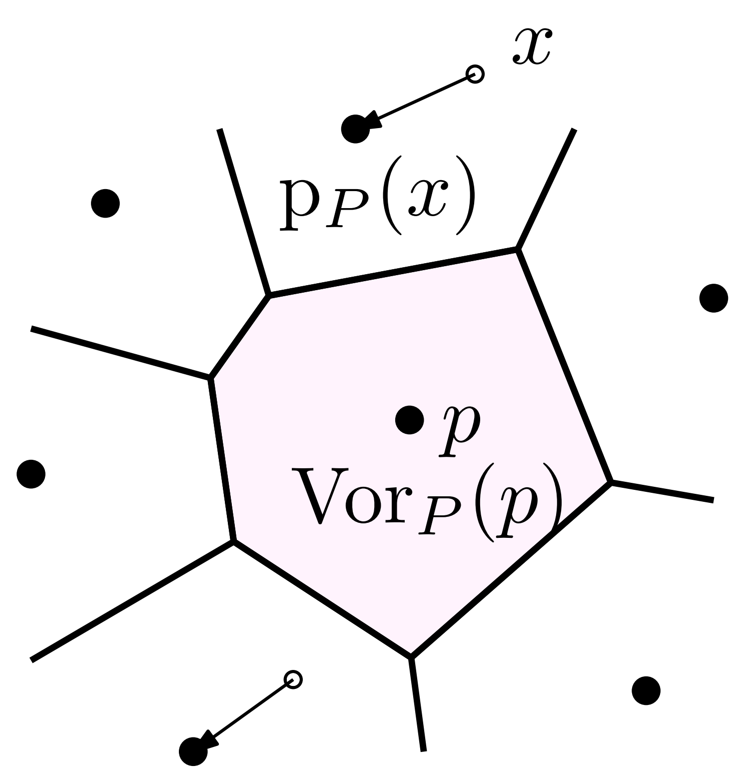

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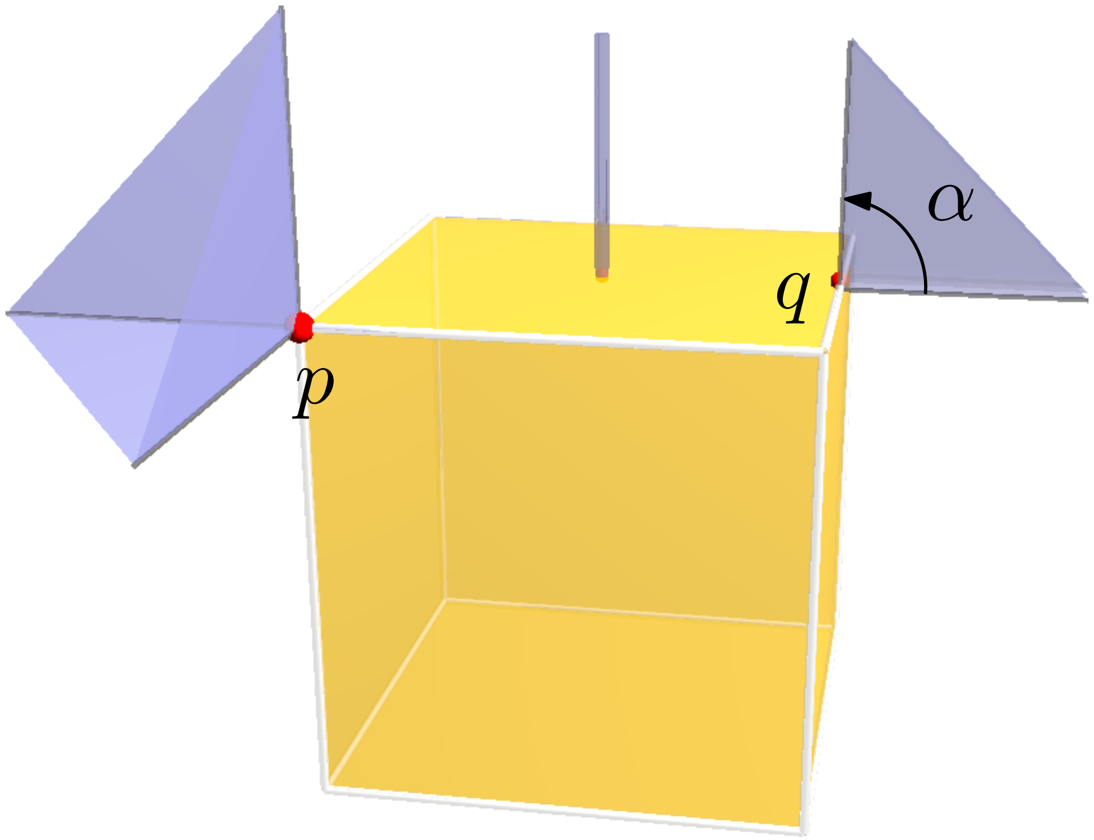
local Minkowski tensor

Voronoi covariance measure of a polyhedron



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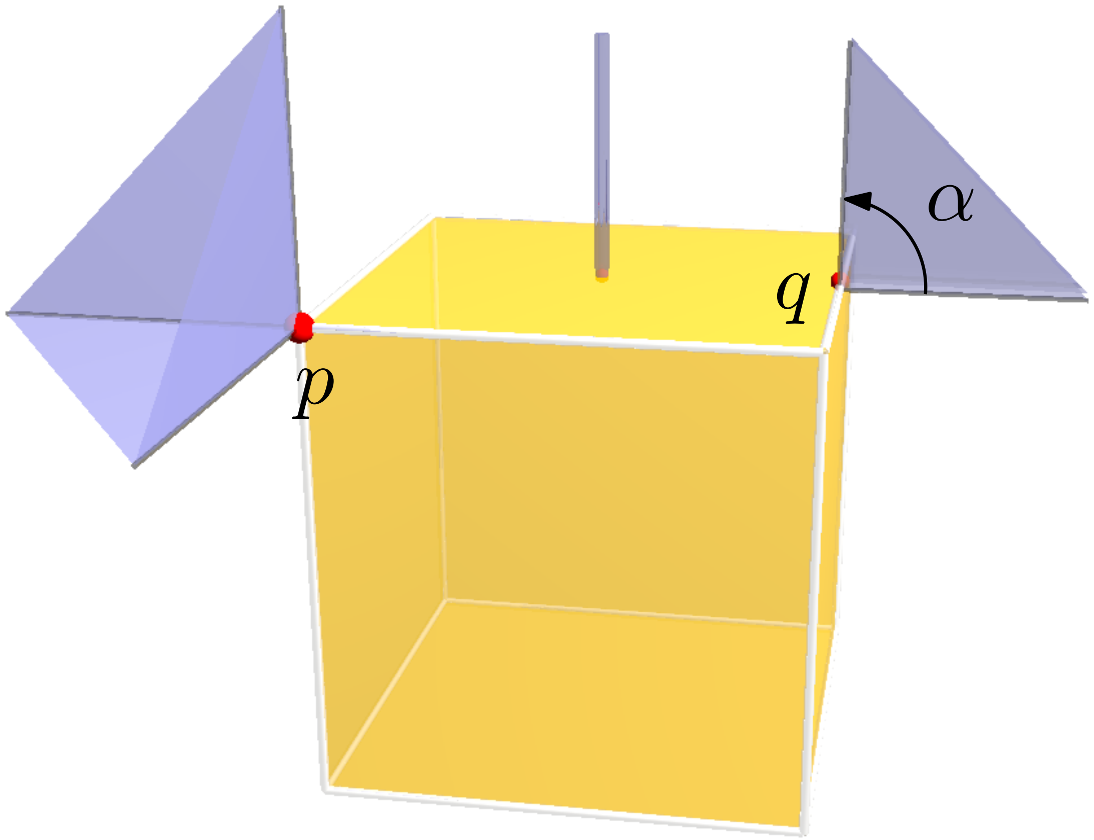

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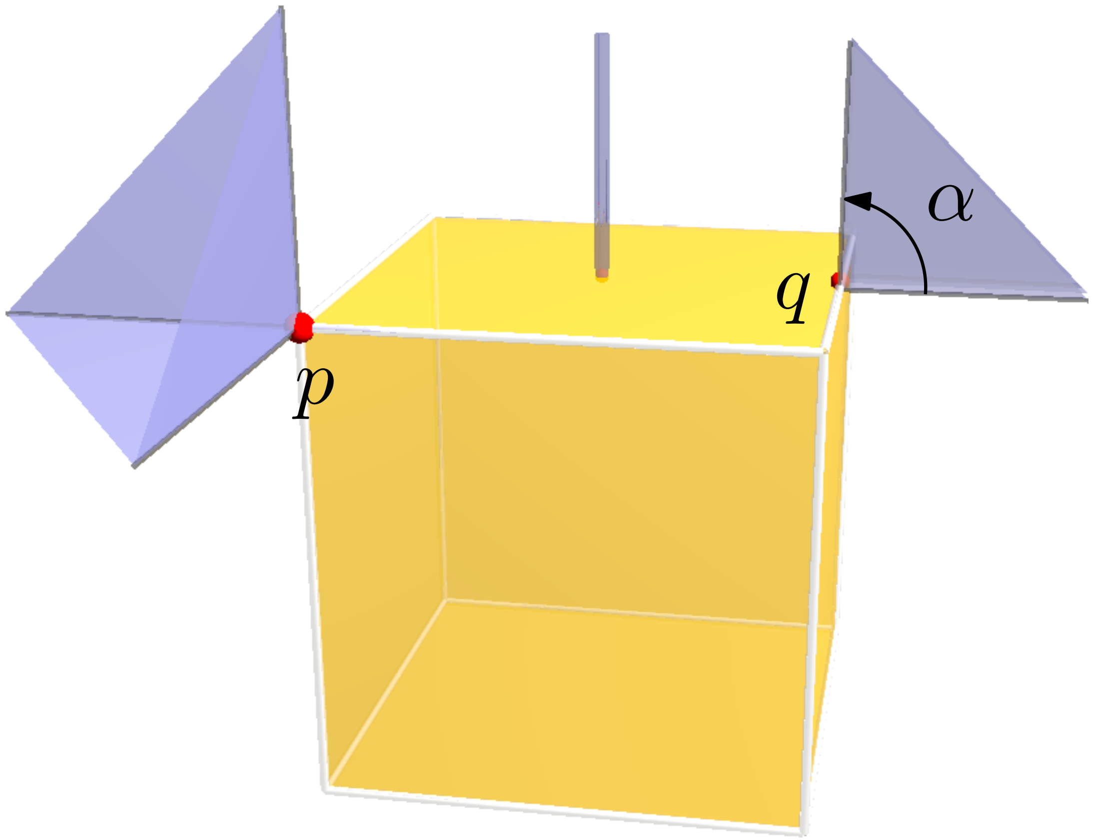

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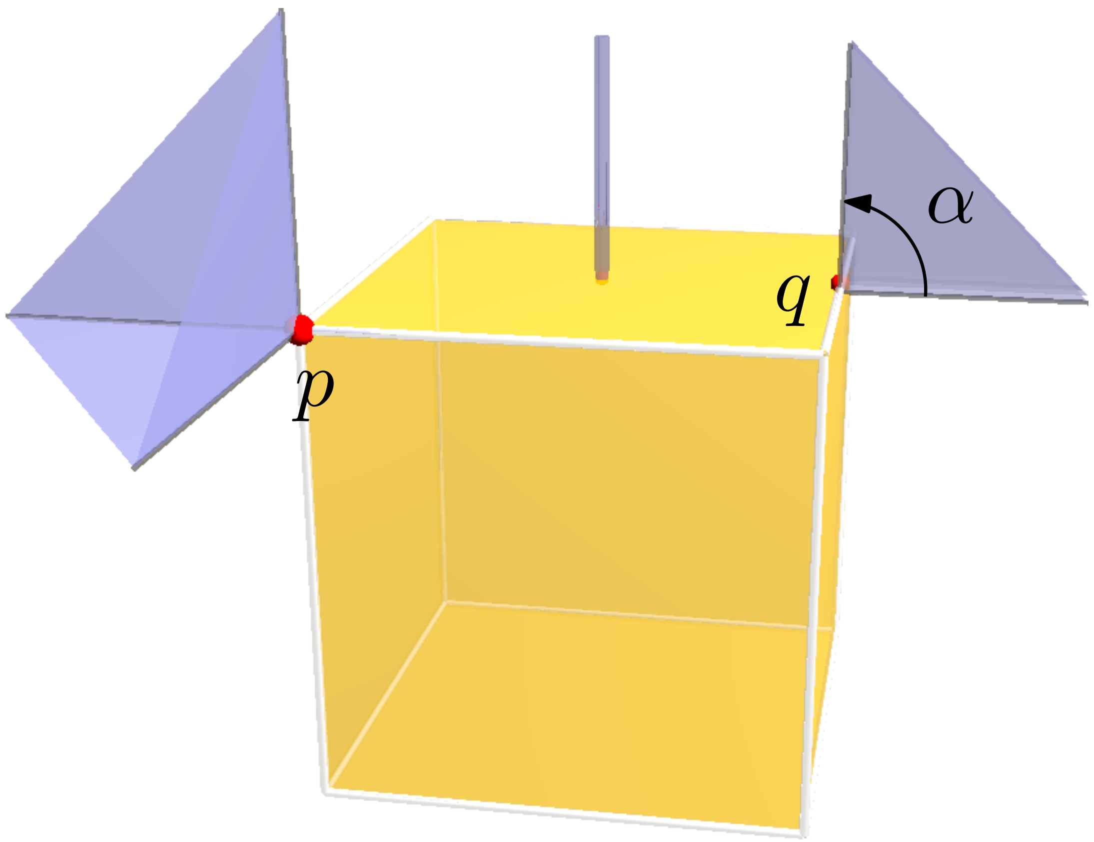

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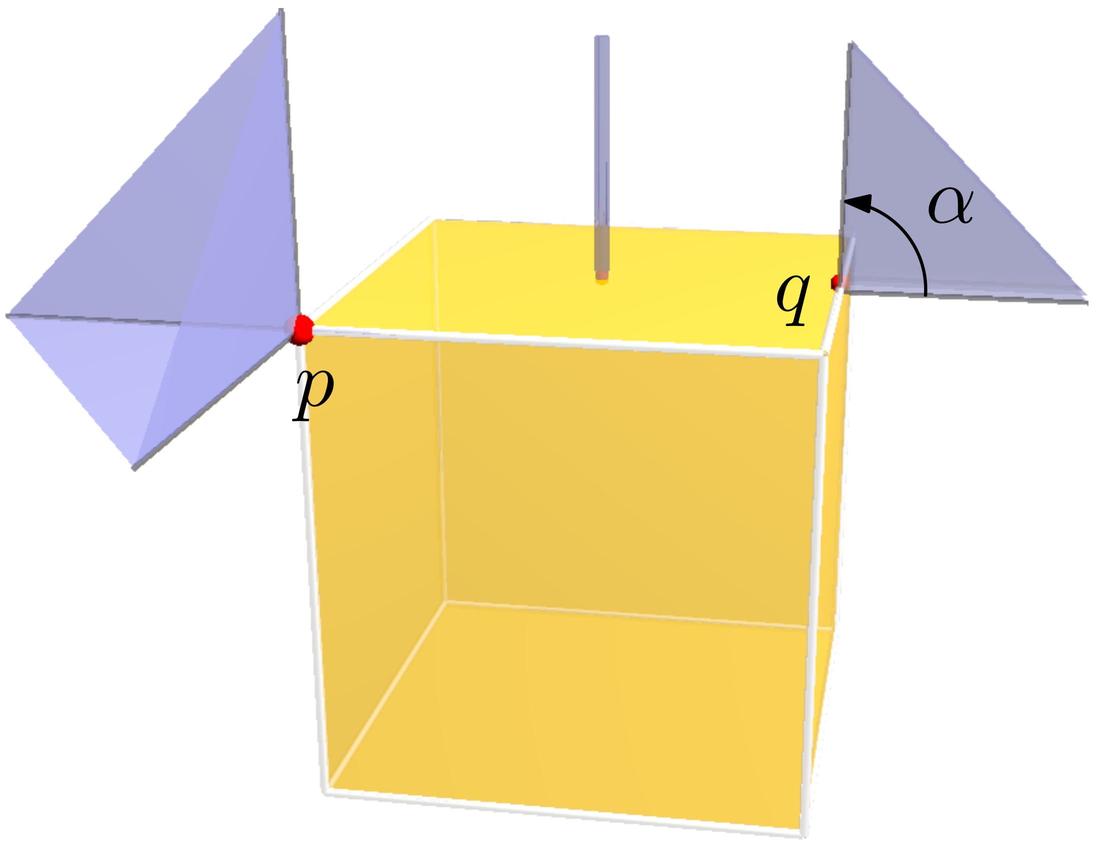

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As $r \to 0$, e_3 converges to the tangent direction of the edge.

Stability of the Voronoi covariance measure

Bounded-Lipschitz distance for tensor-valued measures μ, ν

$$\mathrm{d}_{\mathrm{bL}}(\mu, \nu) := \sup_{f \in \mathrm{BL}_1} \| \int f \,\mathrm{d}\, \mu - \int f \,\mathrm{d}\, \nu \|_{\mathrm{op}}$$

where for $A \in \mathrm{Sym}^+(\mathbb{R}^d)$, $\|A\|_{\mathrm{op}} = \sup_{v \in \mathbb{R}^d \setminus 0} \langle Av | v \rangle / \|v\|^2$

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Theorem: Let $K, L \subseteq \mathbb{R}^d$ be compact and E a bounded domain

$$\mathrm{d_{bL}}(\mathcal{V}_{K,E}, \mathcal{V}_{L,E}) \leq c_{E,K} \sqrt{\mathrm{d_H}(K, L)}$$

assuming that $\mathrm{d_H}(K, L) \leq \mathrm{diam}(K)$.

[M., Ovsjanikov, Guibas 2009]

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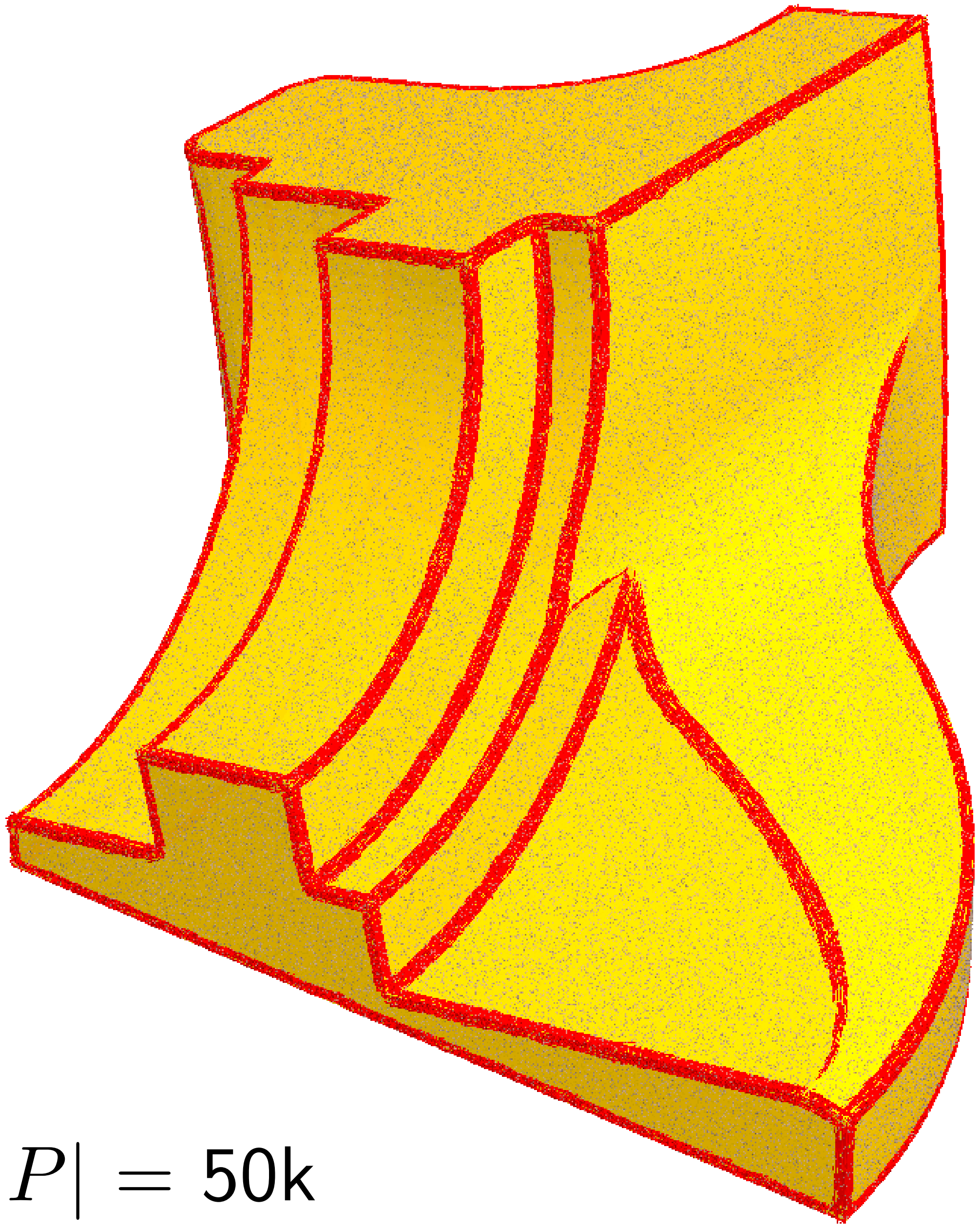
Corollary: Given compact sets K, L with $\mathrm{d_H}(K,L) \le \mathrm{diam}(K)$,

$$\mathrm{d_{bL}}(\mathcal{V}_{K,K^R}, \mathcal{V}_{L,L^R}) \le c_{E,K,R}\sqrt{\mathrm{d_H}(K,L)}$$

$\longrightarrow$ Inference result for local Minkowski tensors of sets with positive reach.

Numerical application of VCM: edge extraction

- $(\lambda_i(p))_{1 \leq i \leq 3} :=$ sorted eigenvalues of $\mathcal{V}_{P,P^R}(\mathrm{B}(p,r))$
- mark p as edge if $\lambda_2(p)/(\lambda_1(p) + \lambda_2(p) + \lambda_3(p)) \leq T$



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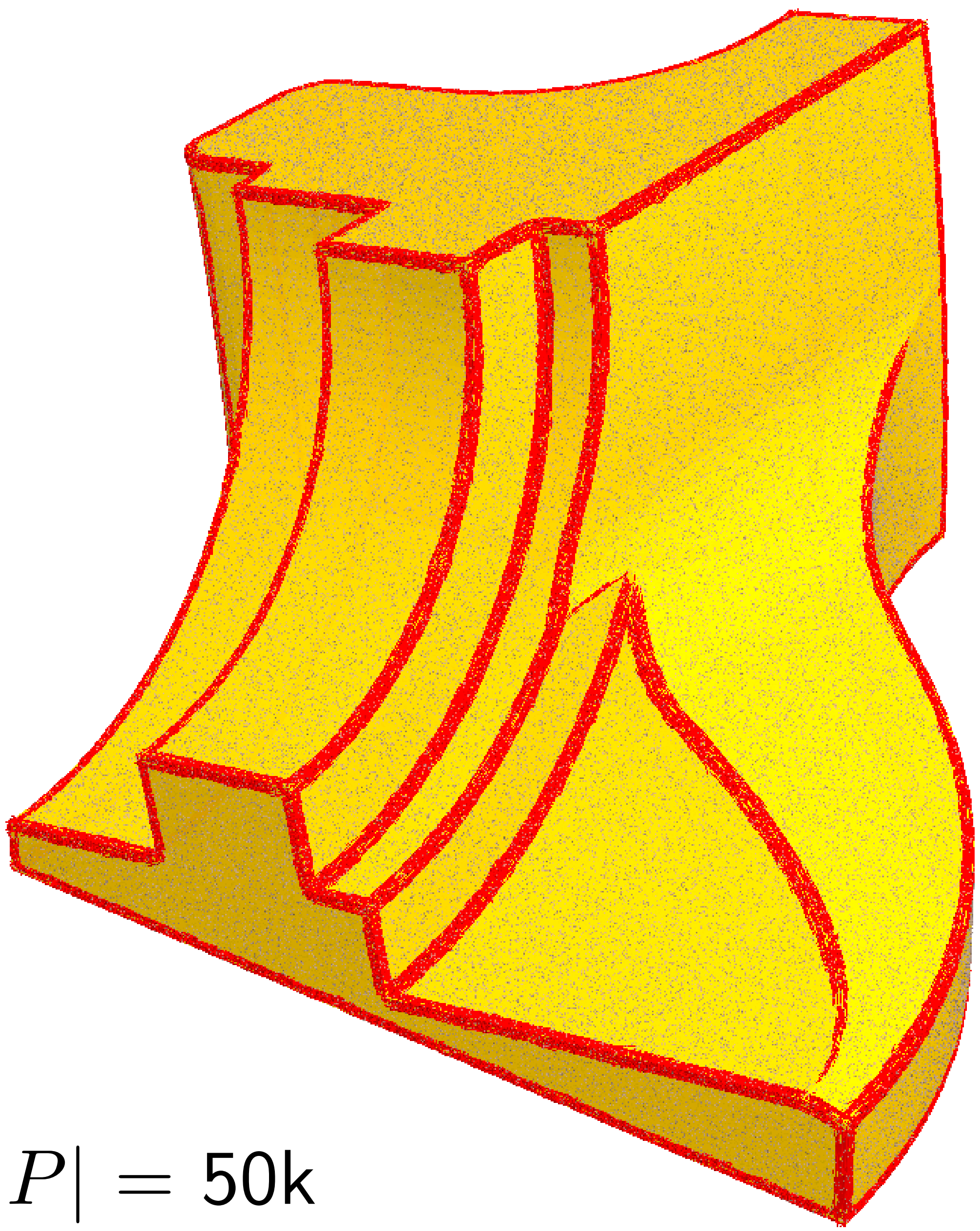

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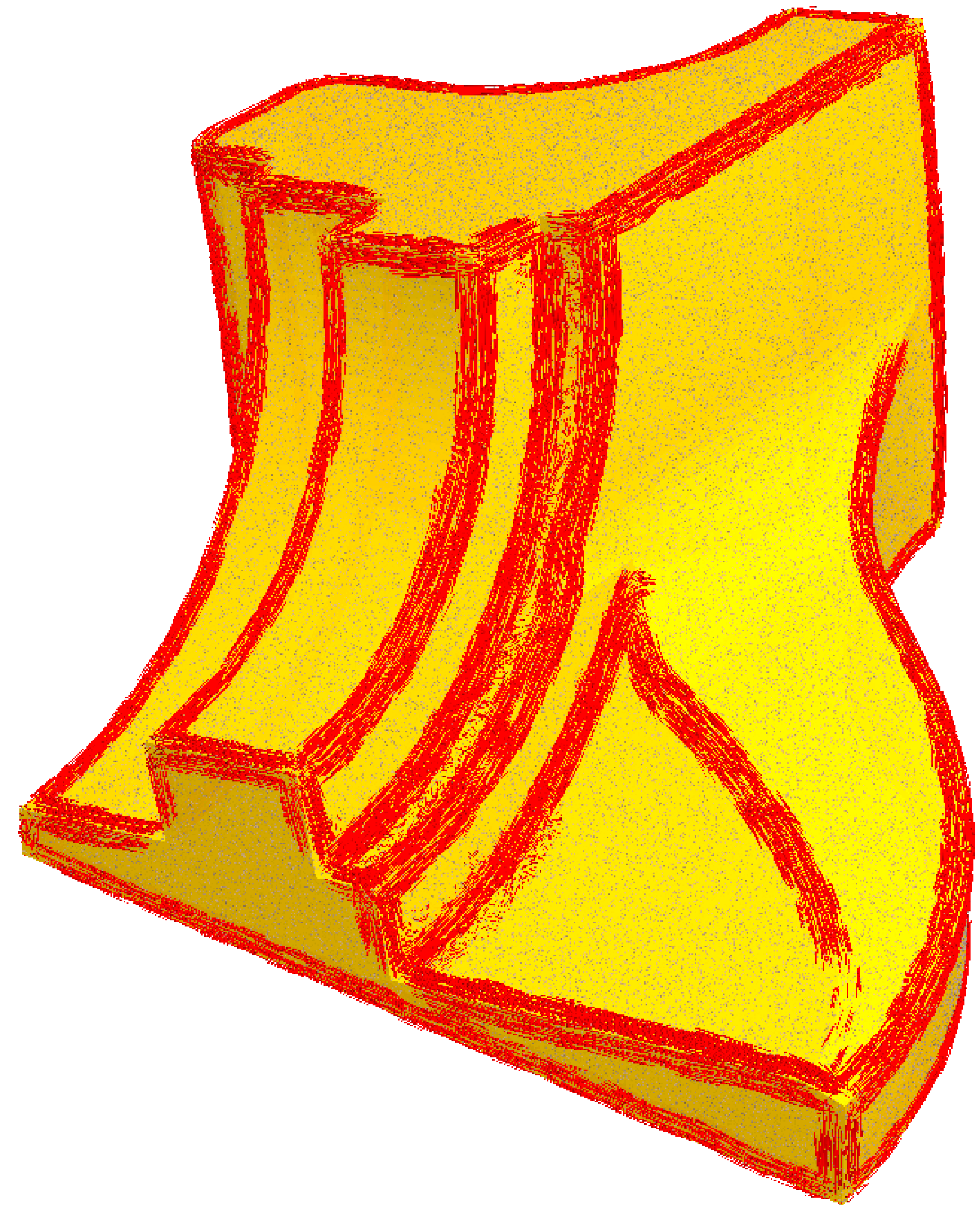

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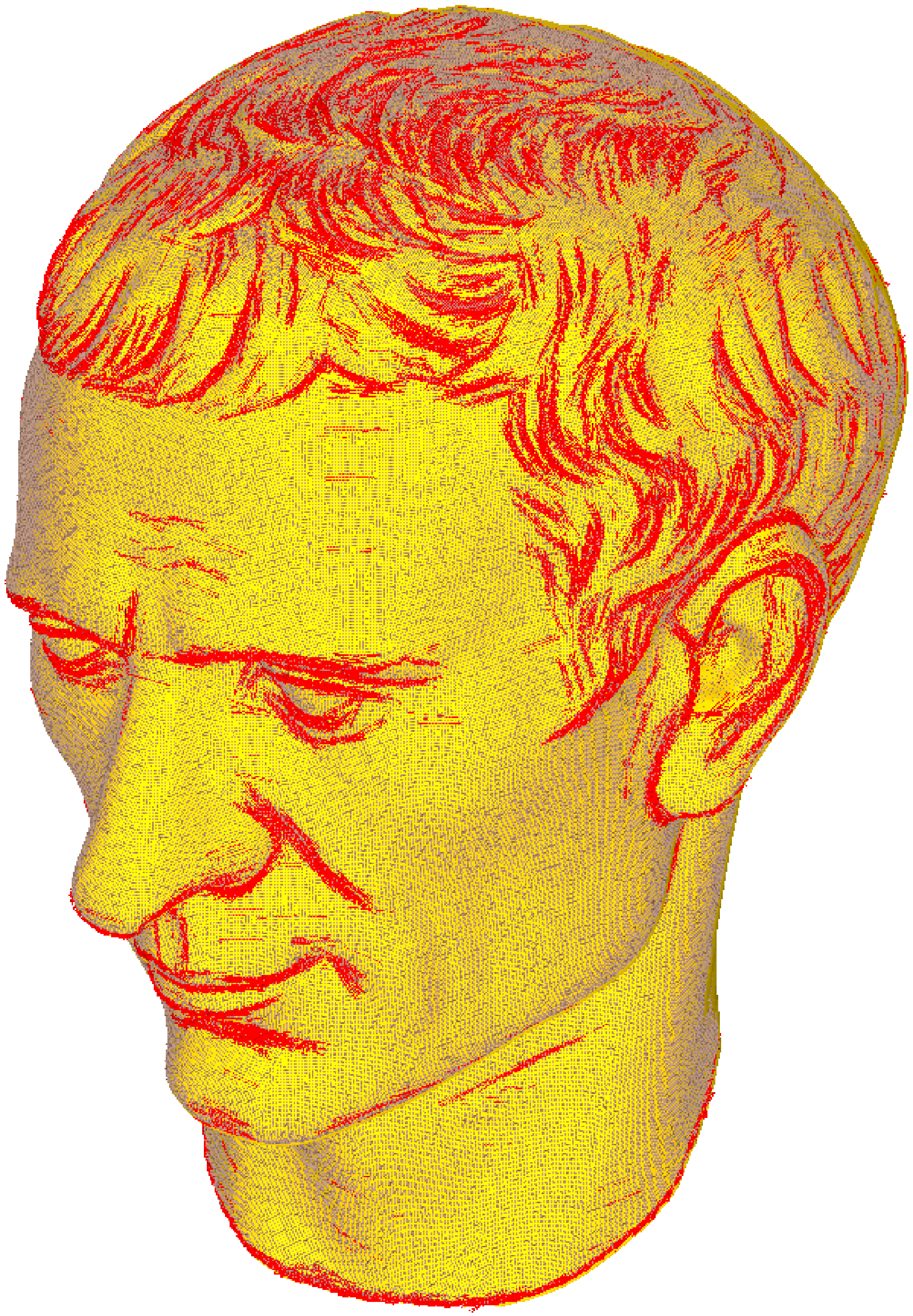
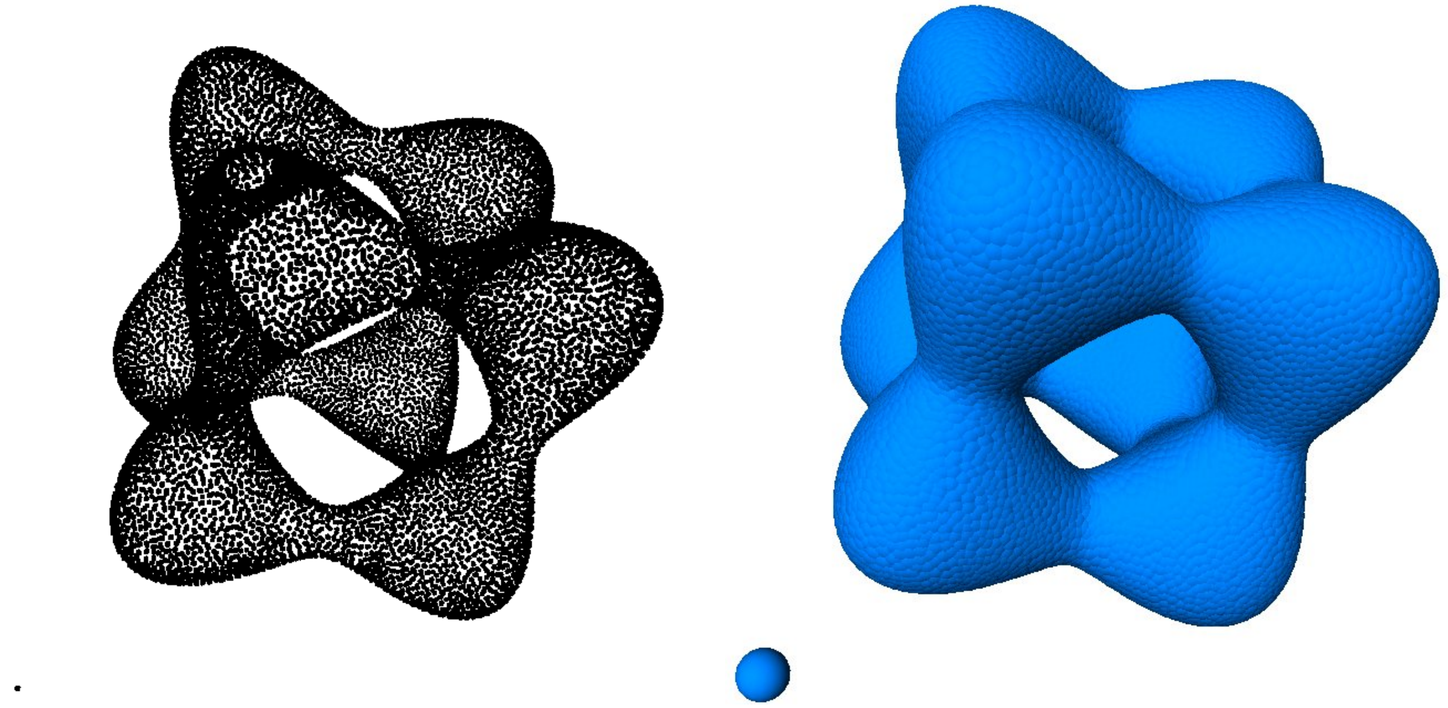


3. Distance to a measure and robust VCM

Distance-like functions

Offset-based inference fails even with a **single** outlier!



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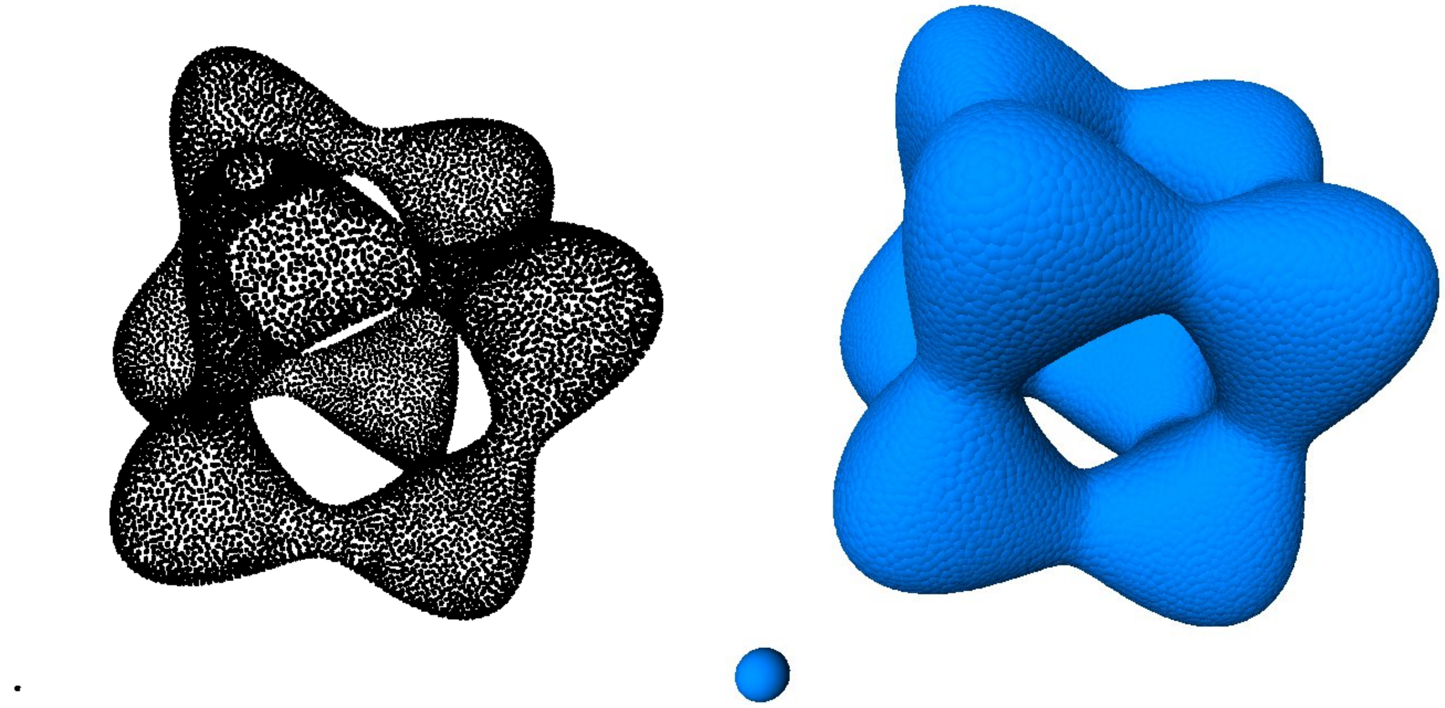

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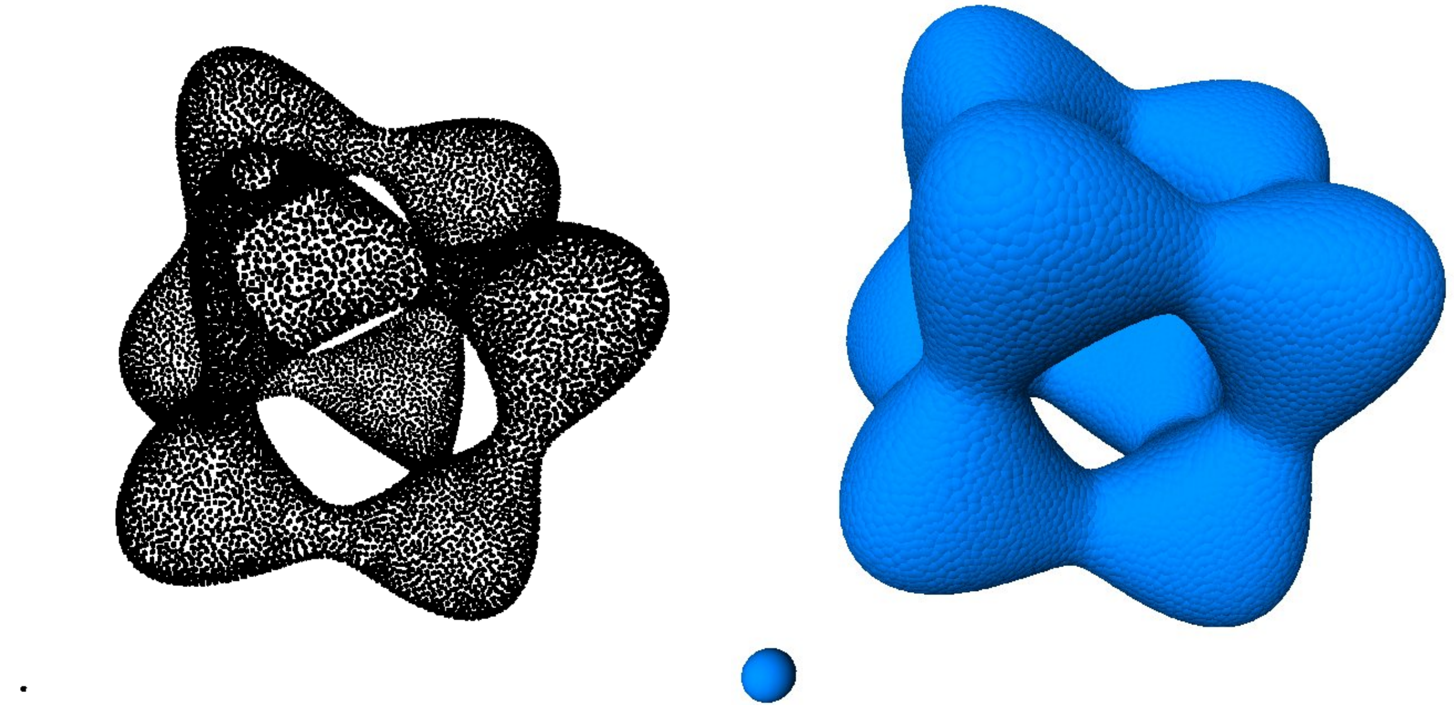

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The stability theorems mentioned before can be generalized to:

$$P, \mathrm{d}_P \longleftrightarrow \phi \text{ distance-like}$$

$$P^r \longleftrightarrow \phi^{-1}([0, r])$$

$$\mathrm{d}_{\mathrm{H}}(P, K) \leq \varepsilon \longleftrightarrow \| \mathrm{d}_K - \phi\|_\infty \leq \varepsilon$$

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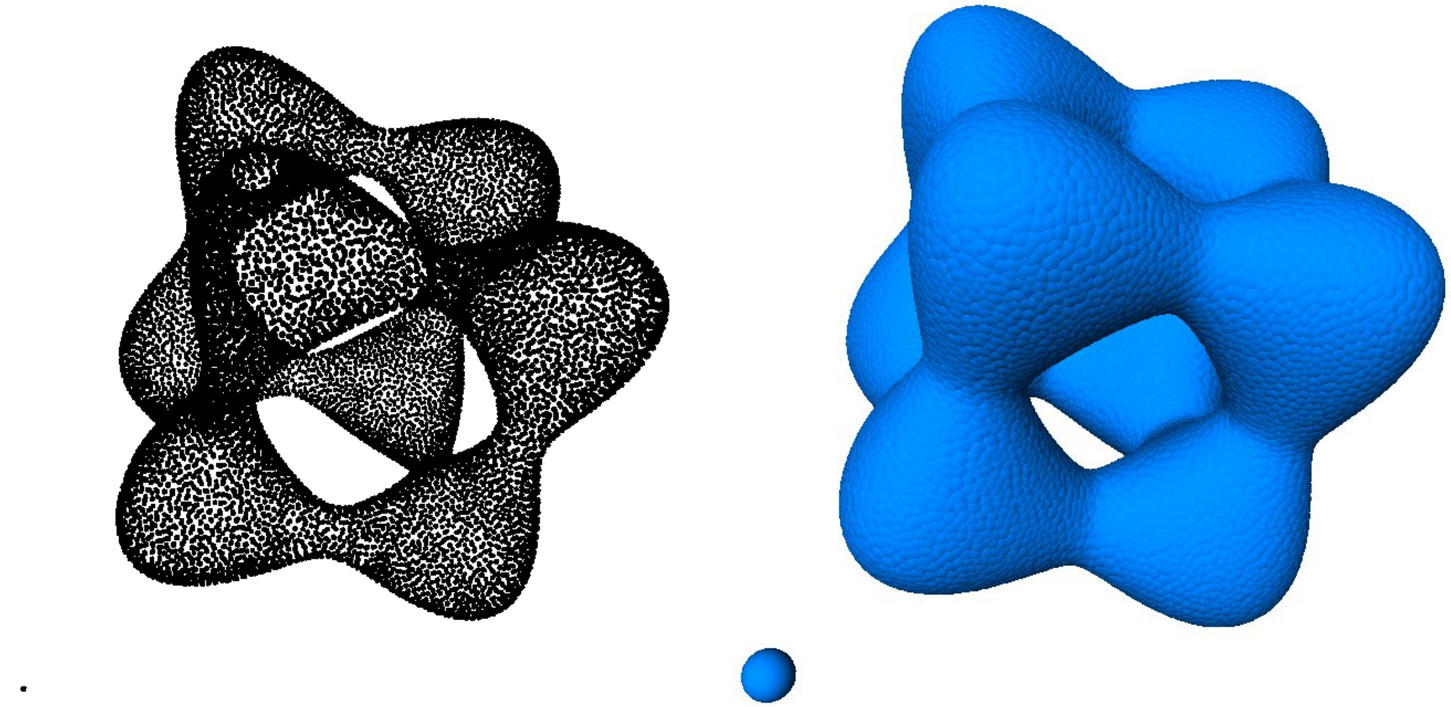

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Idea: Replace d_P with a distance-like function more resilient to outliers.

Generalized Voronoi covariance measure

The **Voronoi covariance measure** of a distance-like function ϕ is a tensor-valued measure on $\mathbb{R}^d$. For $B \subseteq \mathbb{R}^d$,

$$\mathcal{V}_{\phi,E}(B) = \int_E \mathbf{n}_\phi(x) \otimes \mathbf{n}_\phi(x) \mathbf{1}_B(x - \mathbf{n}_\phi(x)) \,\mathrm{d}\,\mathcal{H}^d(x)$$

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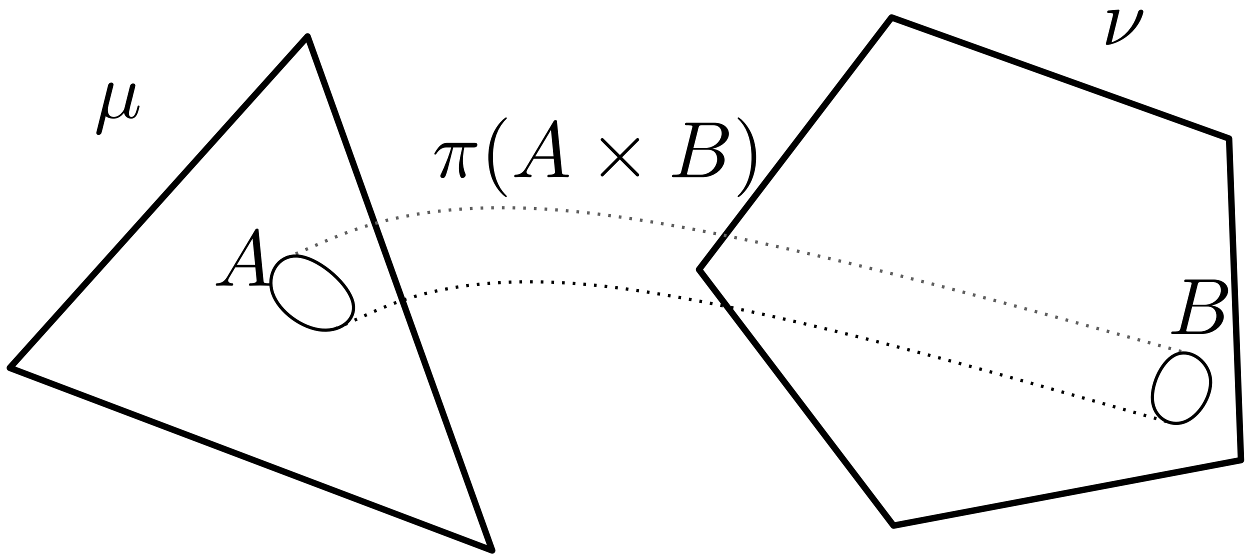
[Cuel, Lachaud, M., Thibert 2014]

Wasserstein distance



μ, ν non-negative measures,
$\mu(\mathbb{R}^d) = \nu(\mathbb{R}^d)$

Transport plan: non-negative measure π on $\mathbb{R}^d \times \mathbb{R}^d$ s.t.

$$\pi(A \times \mathbb{R}^d) = \mu(A)$$
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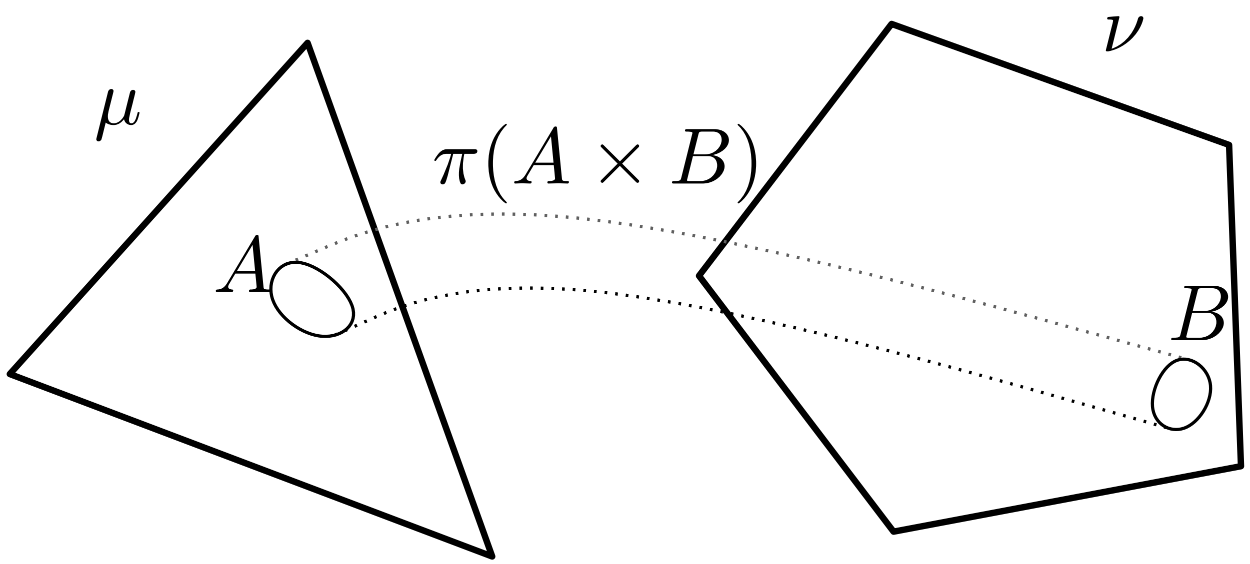

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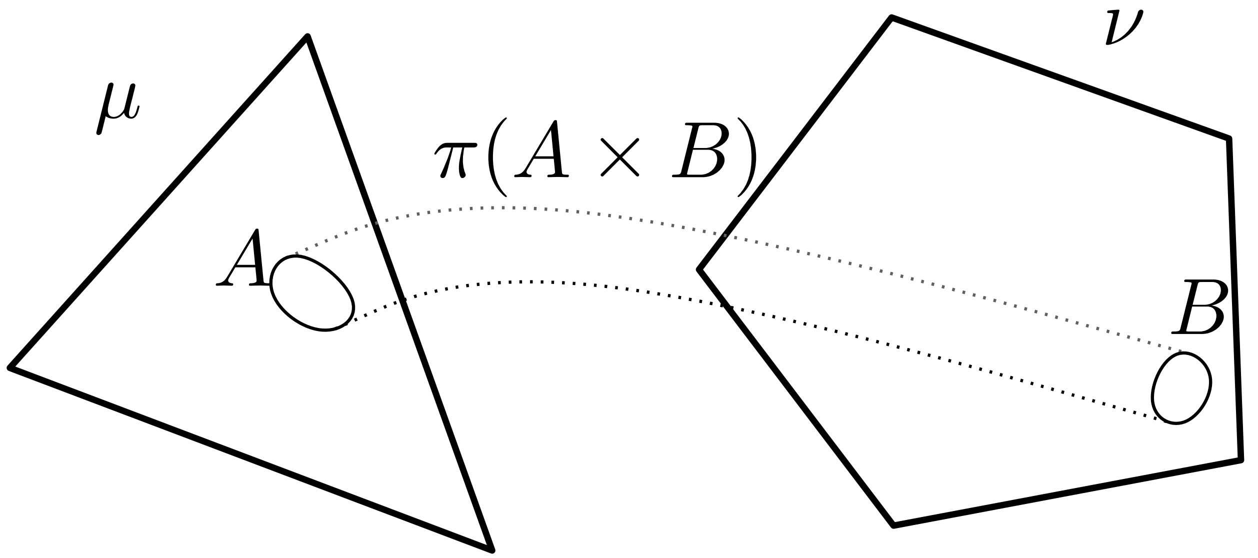

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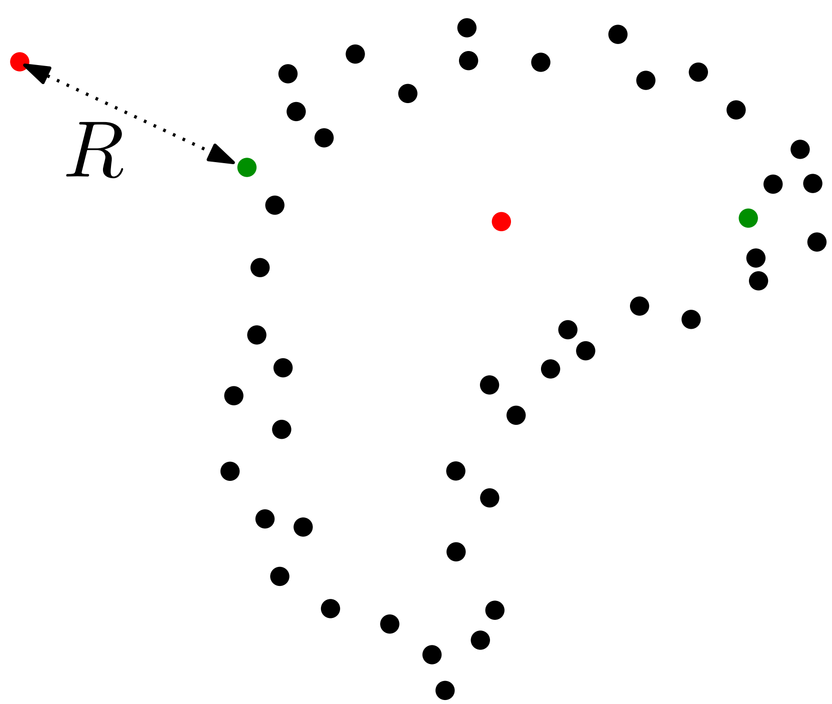
Example: point cloud P $\longrightarrow$ measure $\mu_P := \frac{1}{d} \sum_{p \in P} \delta_p$



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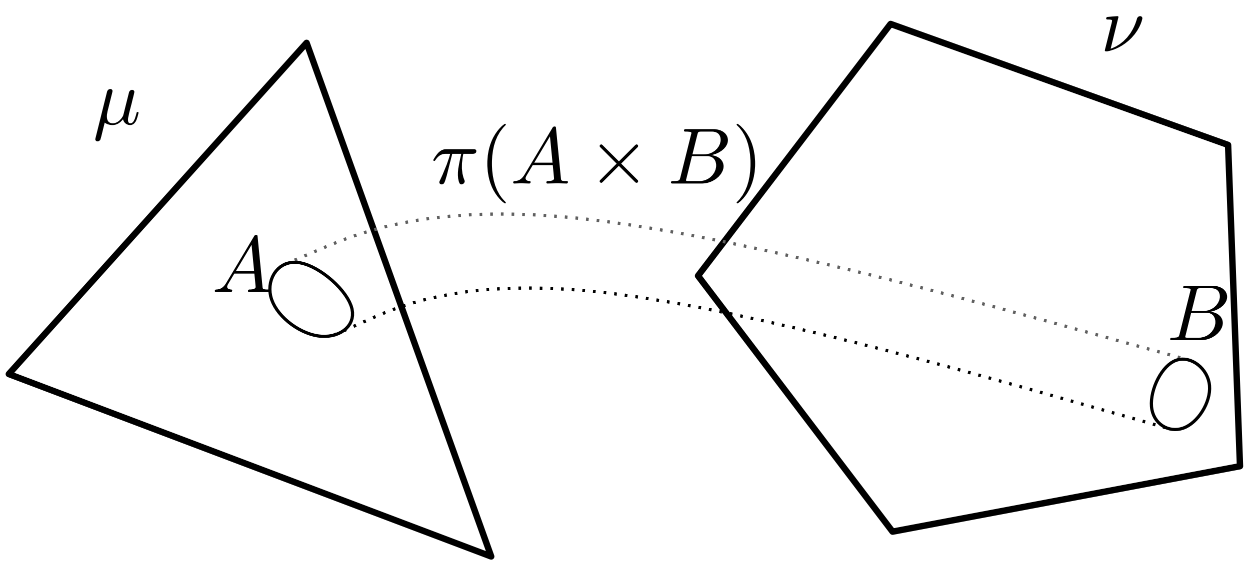

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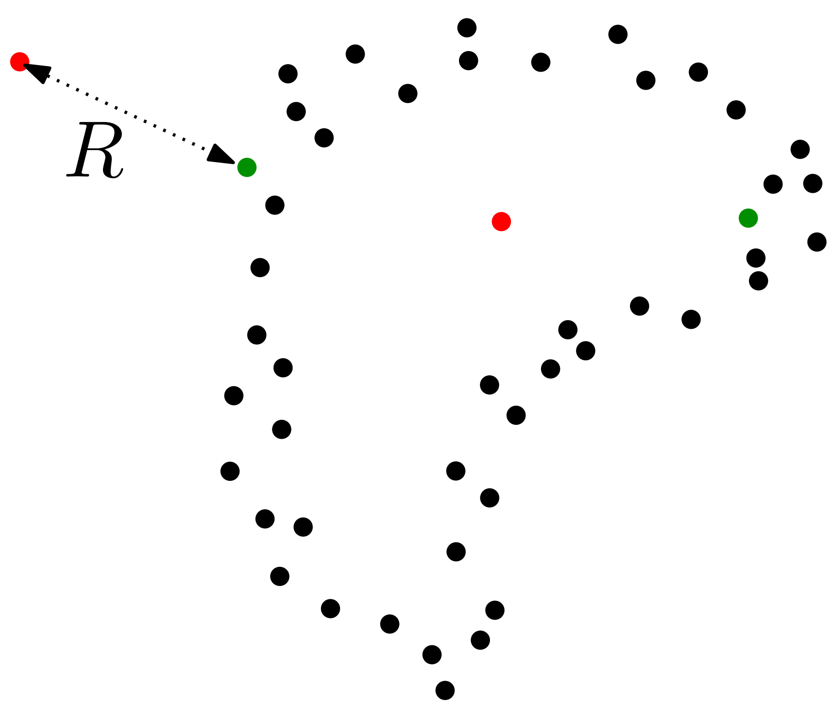
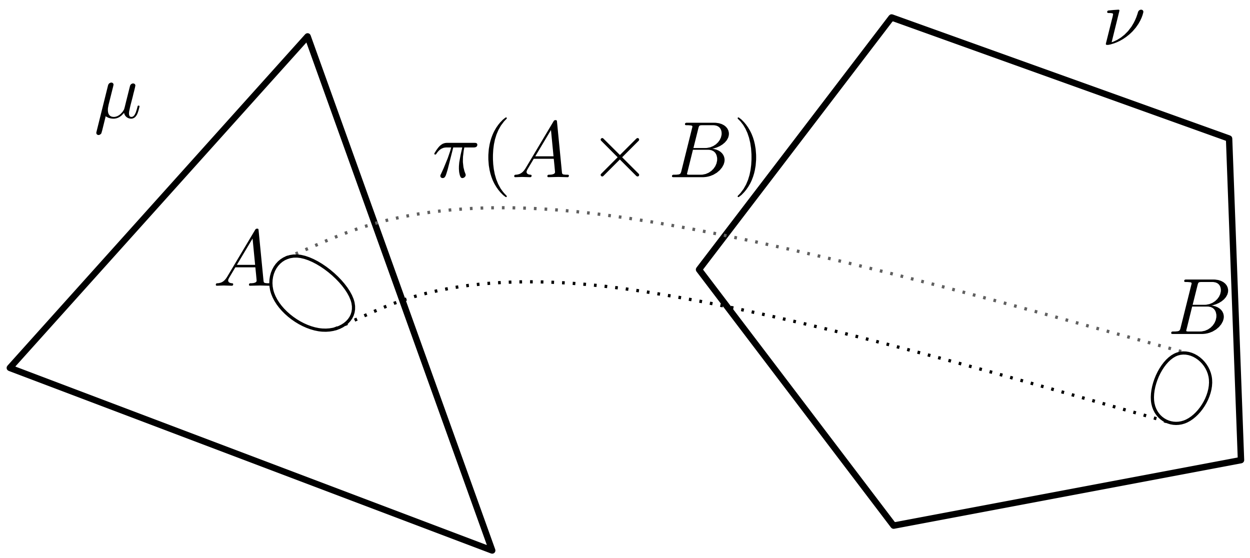


if $P = \bullet \cup \bullet$ and $Q = \bullet \cup \bullet$

then $\mathrm{d_H}(P, Q) = R$ and $\mathrm{W}_2(\mu_P, \mu_Q) \leq \frac{k}{N} R$

In practice, $\mathrm{W}_2(\mu_P, \mu_Q) \ll \mathrm{d_H}(P, Q)$

Wasserstein distance



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Summary:

(compact sets, $\mathrm{d_H}$)	(probability measures, W_2)
K, d_K	$\mu, \mathrm{d}_{\mu,m}$
d_K distance-like	$\mathrm{d}_{\mu,m}$ distance-like
$\\| \mathrm{d}_K - \mathrm{d}_{K'} \\| \leq \mathrm{d_H}(K, K')$	$\\| \mathrm{d}_{\mu,m} - \mathrm{d}_{\mu',m} \\|_\infty \leq m^{-1/2}\, \mathrm{W}_2(\mu, \mu')$

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[Chazal-Cohen-Steiner-M '09]

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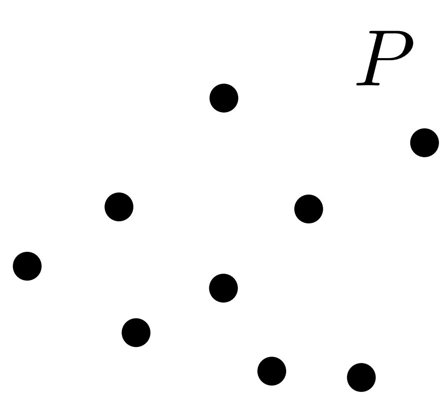

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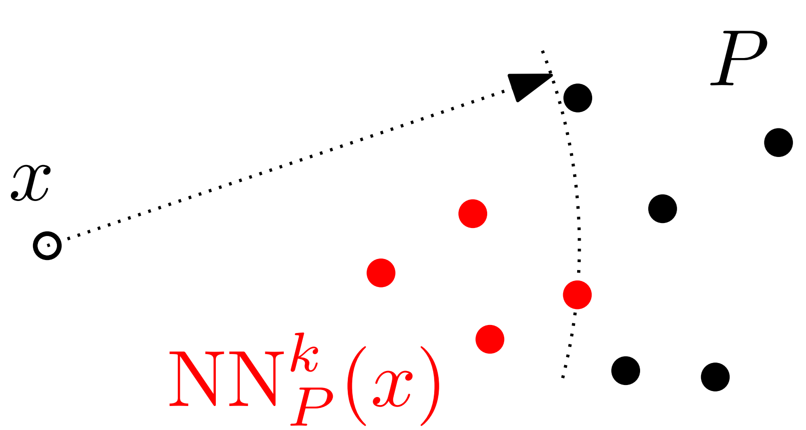

$$\mathrm{d}^2_{\mu_P,m} = \frac{1}{k} \sum_{p \in \mathrm{NN}_P^k(x)} \|x-p\|^2$$

where $\mathrm{NN}_P^k(x) = k$ nearest neighbors of x in P

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(*) $\mathrm{d}_{\mu,m}^2(x) = \min_{\nu \in \mathrm{Sub}_m(\mu)} m \int_{\mathbb{R}^d} \|x - p\|^2 \,\mathrm{d}\,\nu(p)$

Semiconcavity of the distance to a measure

Proposition: The function $\mathrm{d}_{\mu,m}$ is distance-like.

Proof: We show that $\mathrm{d}^2_{\mu,m} - \|.\|^2$ is concave:

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$$= \min_{\nu \in \mathrm{Sub}_m(\mu)} m \int_{\mathbb{R}^d} (\|x\|^2 + \|p\|^2 - 2\langle x|p\rangle) \, \mathrm{d}\, \nu(p)$$

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$\Longrightarrow$ $\mathrm{d}_{\mu,m}(x)^2 - \|.\|^2$ is concave, and with $\nu :=$ minimizer in (1),

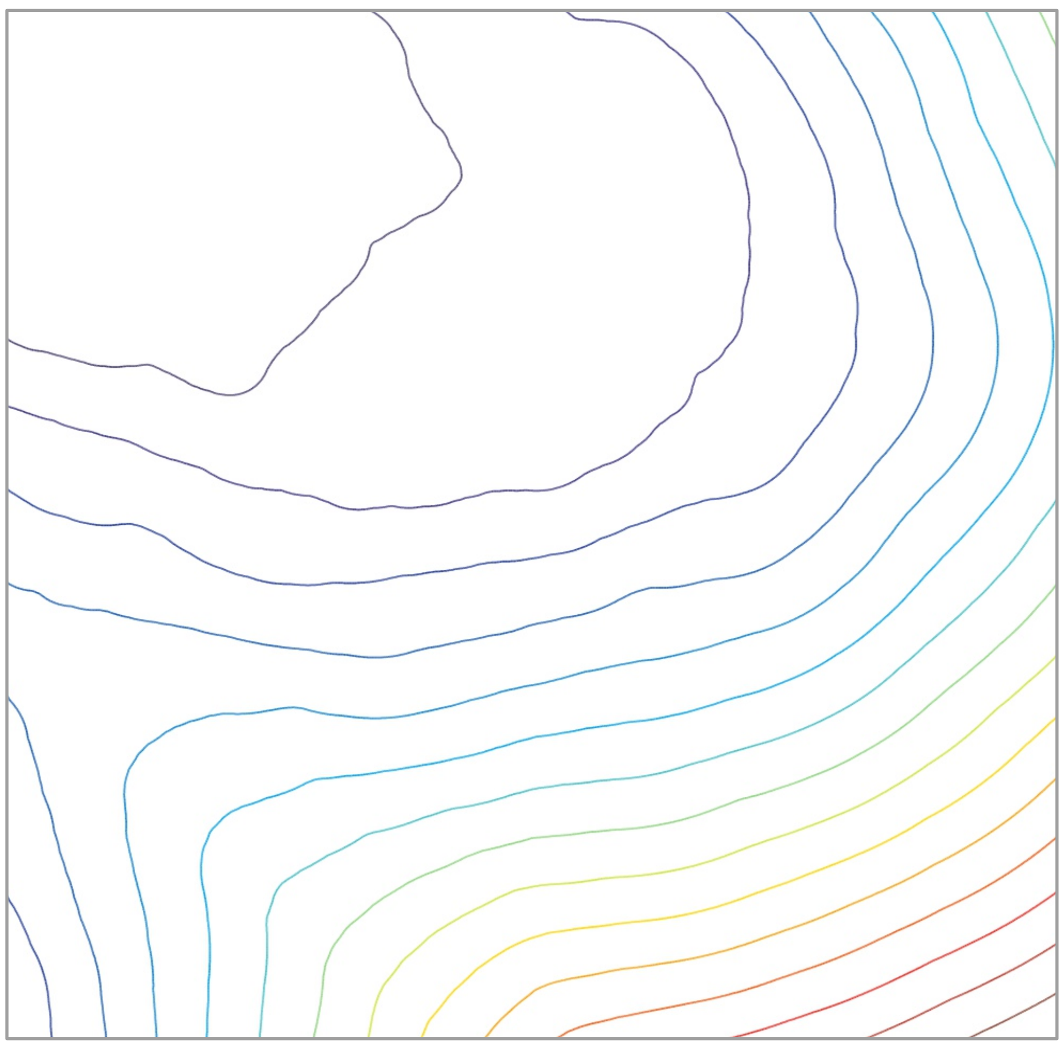
$$\frac{1}{2}\nabla\,\mathrm{d}^2_{\mu,m}(x) = x - m \int_{\mathbb{R}^d} p \,\mathrm{d}\,\nu(p)$$

$$= x - \mathrm{centroid}(\nu)$$

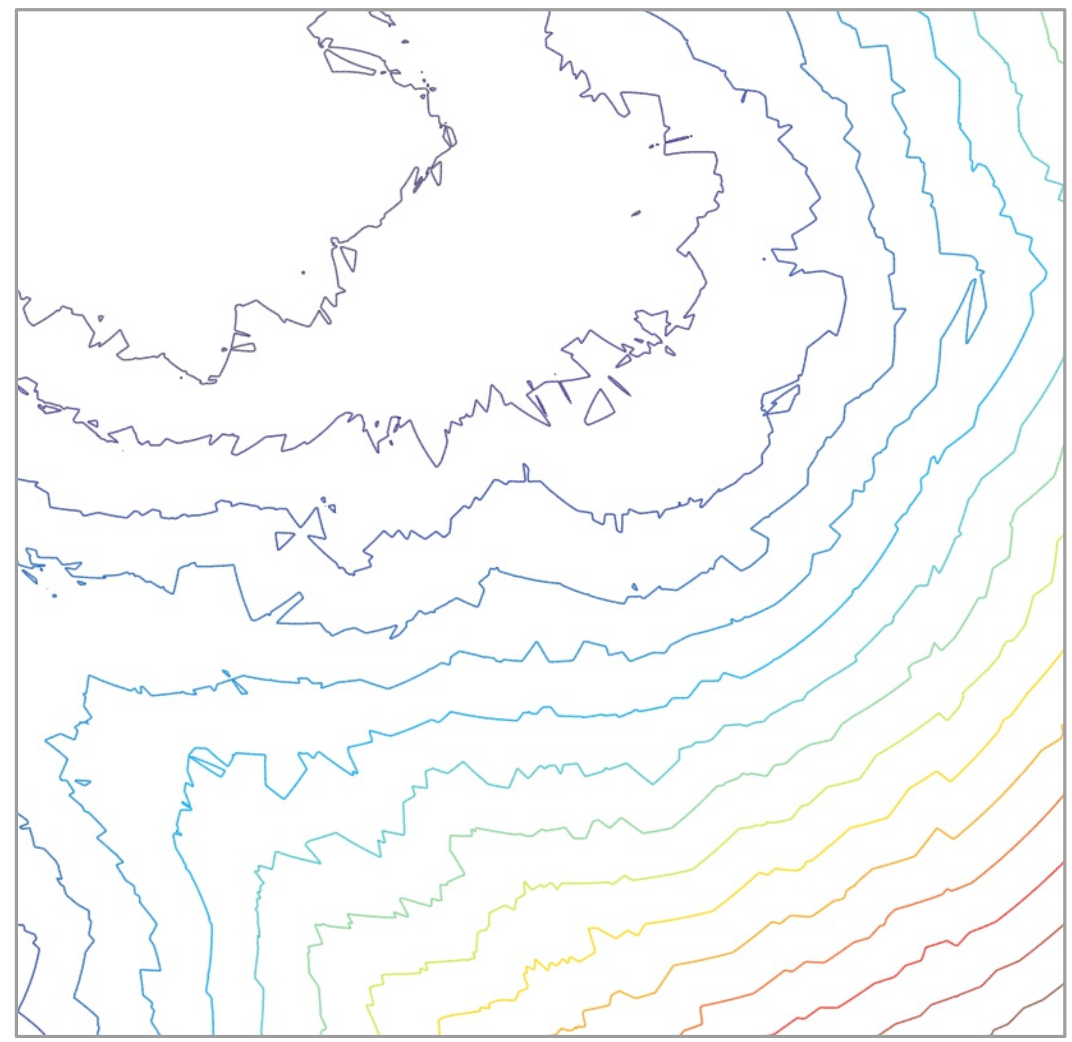
Semiconcavity of the distance to a measure

Proposition: The function $\mathrm{d}_{\mu,m}$ is distance-like.

Illustration: P sampled from a mixture of two Gaussians in $\mathbb{R}^2$.
$|P| = 500$ and $m = 20/500$



$\mathrm{d}_{\mu,m}$



distance to the 20th nearest neighbor

Stability of the distance to a measure

Proposition: $\| \mathrm{d}_{\mu,m} - \mathrm{d}_{\mu',m} \|_\infty \leq m^{-1/2}\, \mathrm{W}_2(\mu, \mu')$

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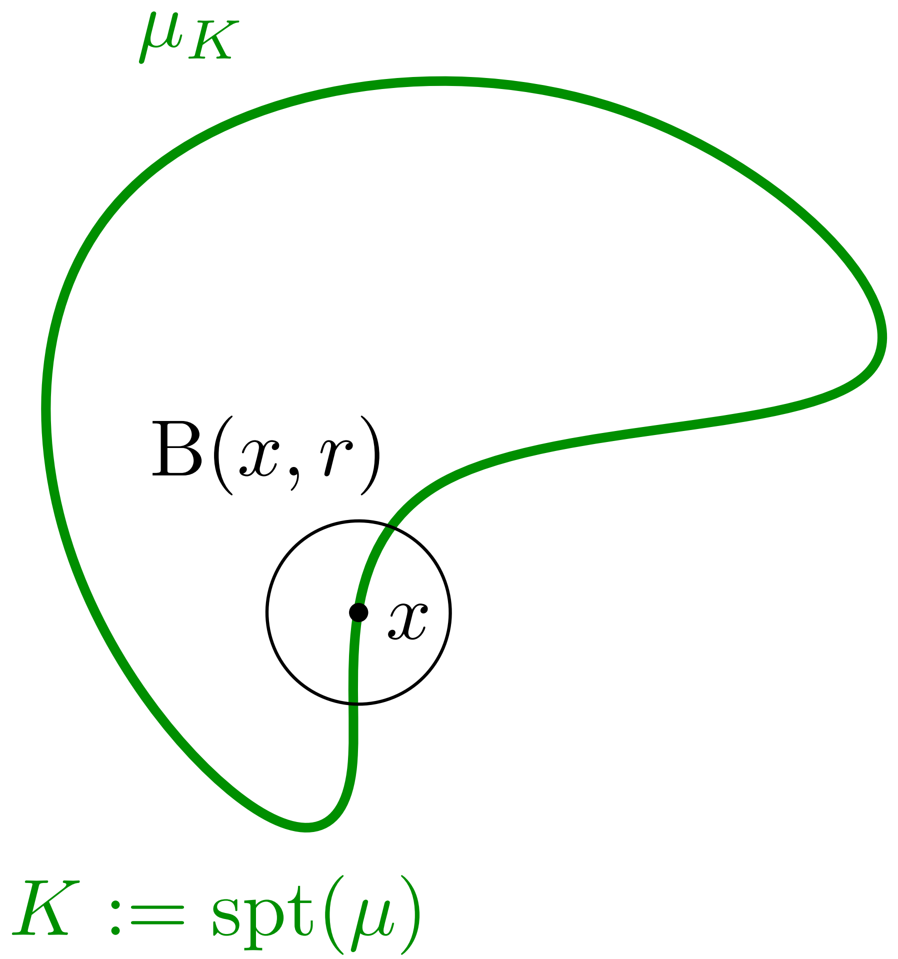

Definition: Assume μ_K is supported on K. Then, $\dim(\mu_K) \geq \ell$ iff $\exists \alpha_K, r_K > 0$ s.t.

$$\forall x \in K, \forall r \leq r_K, \qquad \mu(\mathrm{B}(x,r)) \geq \alpha_K r^\ell$$

Example: Volume measure on a compact surface.

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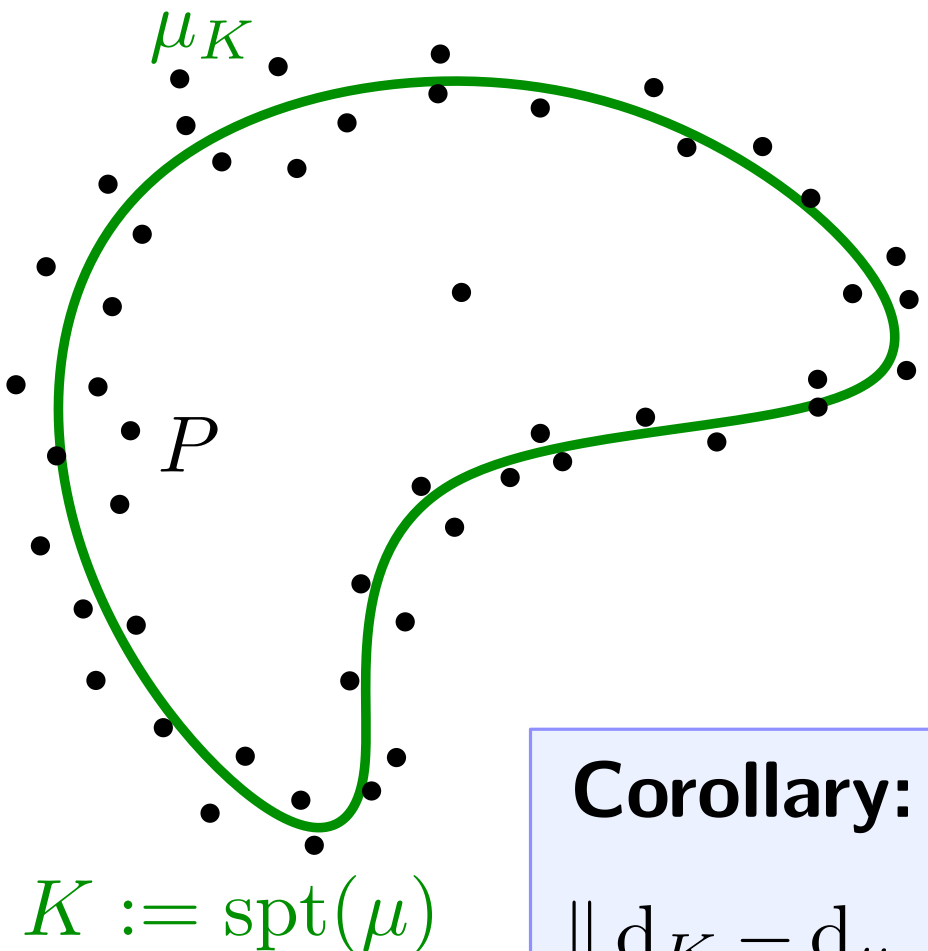

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Corollary: If μ_K has dimension at most ℓ,

$$\| \mathrm{d}_K - \mathrm{d}_{\mu_P,m} \|_\infty \leq \| \mathrm{d}_K - \mathrm{d}_{\mu_K,m} \|_\infty + \| \mathrm{d}_{\mu_K,m} - \mathrm{d}_{\mu_P,m} \|_\infty$$

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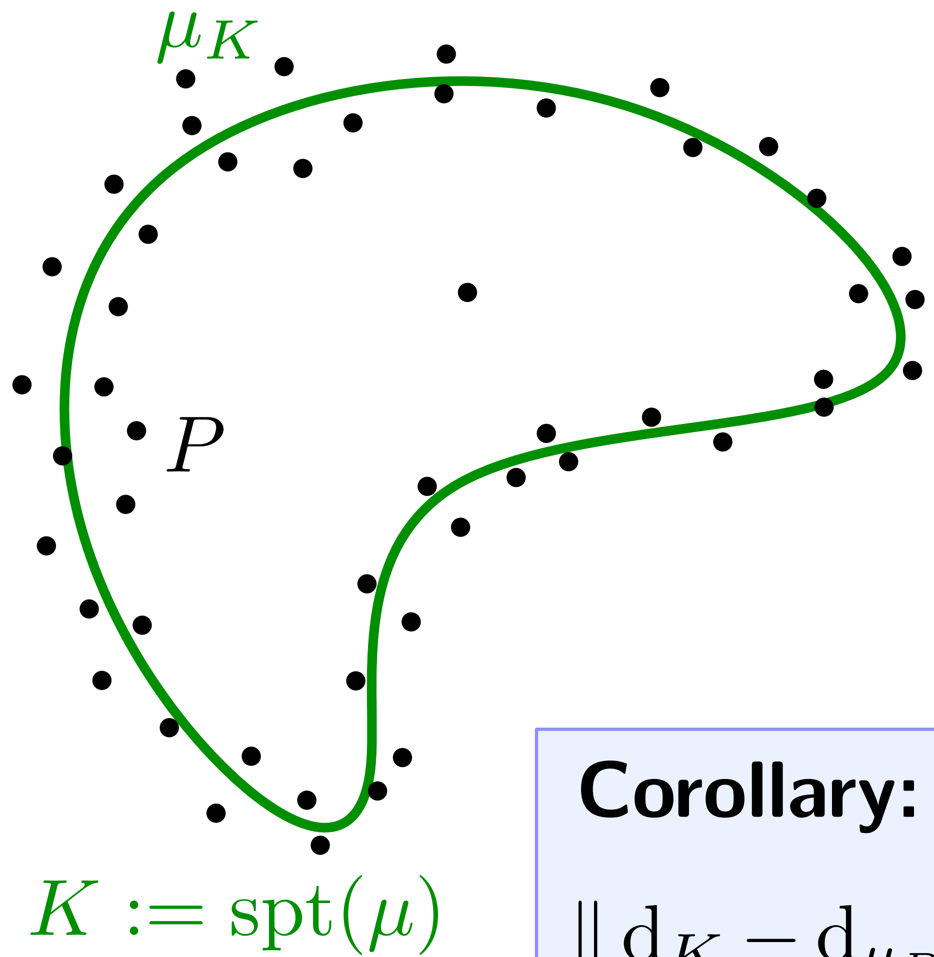

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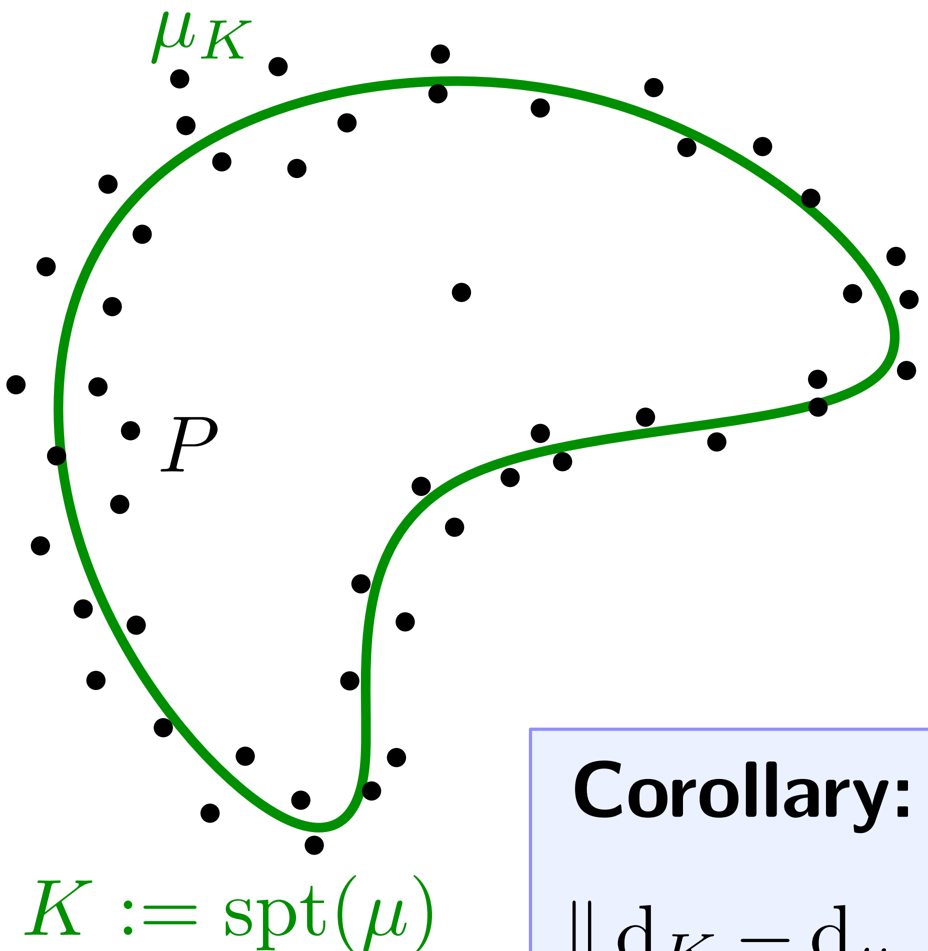

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smoothing — noise

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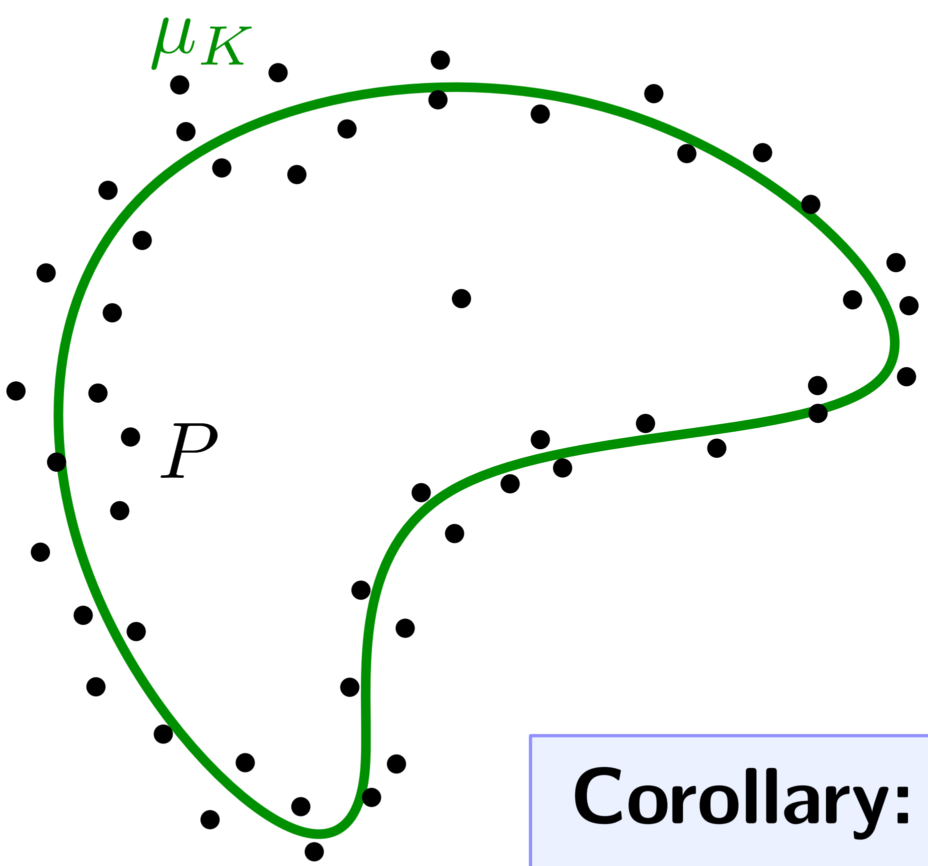

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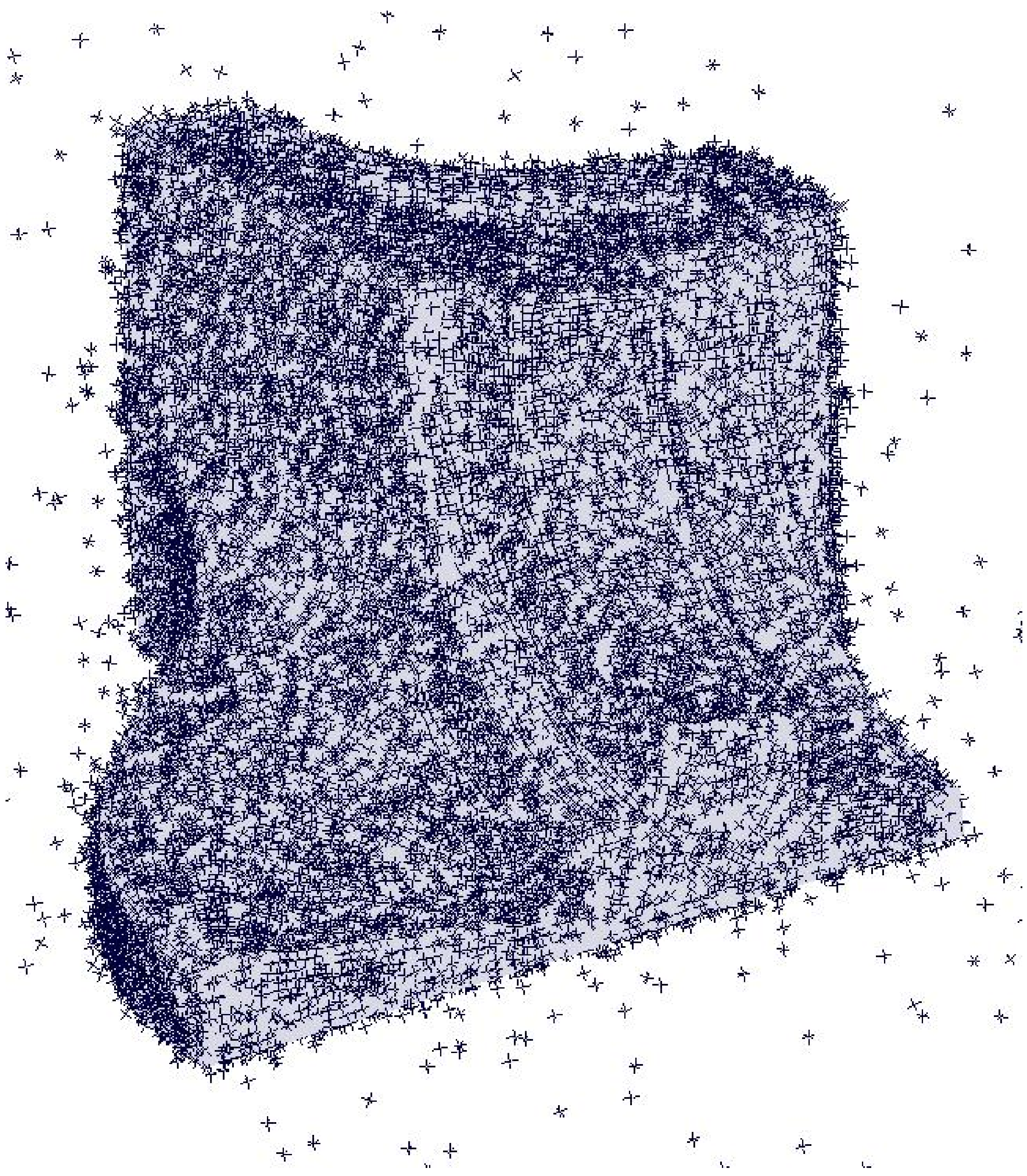
smoothing noise

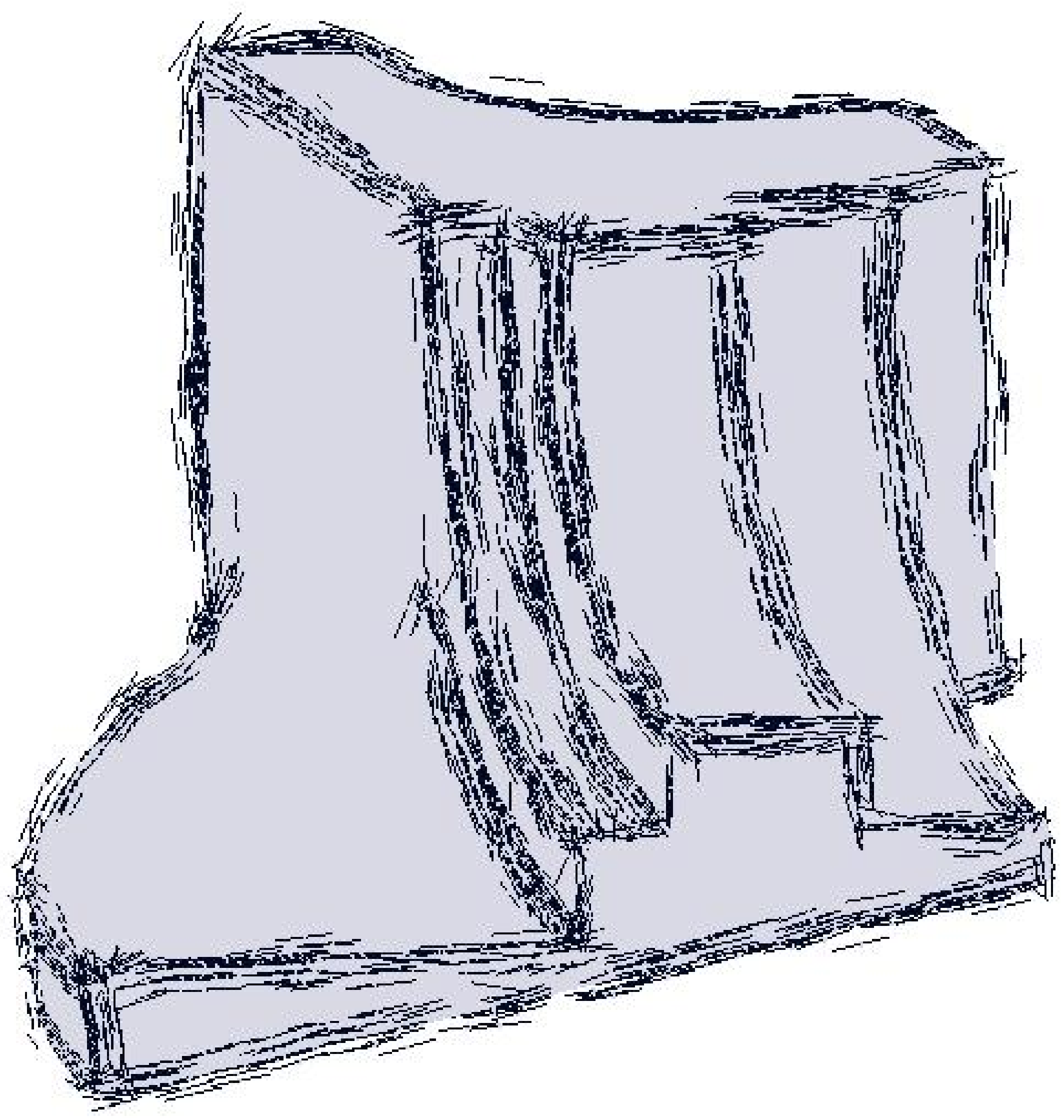
In this case, one can approximate $\mathcal{V}_{K,K^R}$ by $\mathcal{V}_{\phi,\phi^{-1}([0,R])}$ with $\phi = \mathrm{d}_{\mu_P,m}$.

Example: detection of sharp features



Input: 50k point set P

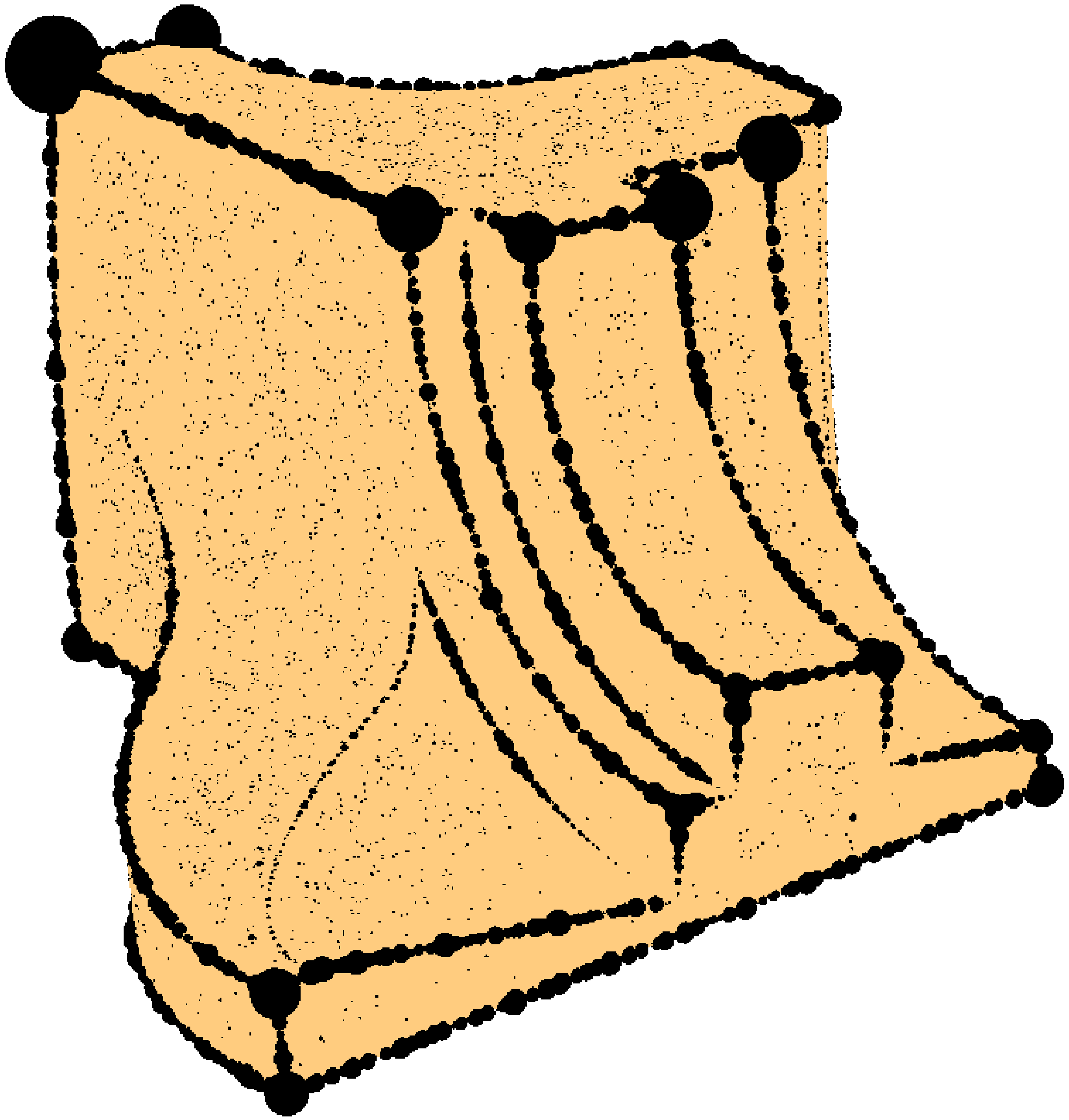
Output: estimated edges and directions



4. Computations

Computation of boundary measures

Boundary measures of a finite sets can be computed via **Monte-Carlo**:



Input: $P \subseteq \mathbb{R}^d$, $r > 0$, $N \in \mathbb{N}$

Output: $\mu_N \sim \mu := \frac{\mu_{P,P^r}}{\mathrm{vol}^d(P^r)}$

$P = 15k$ *points on the "fandisk"*

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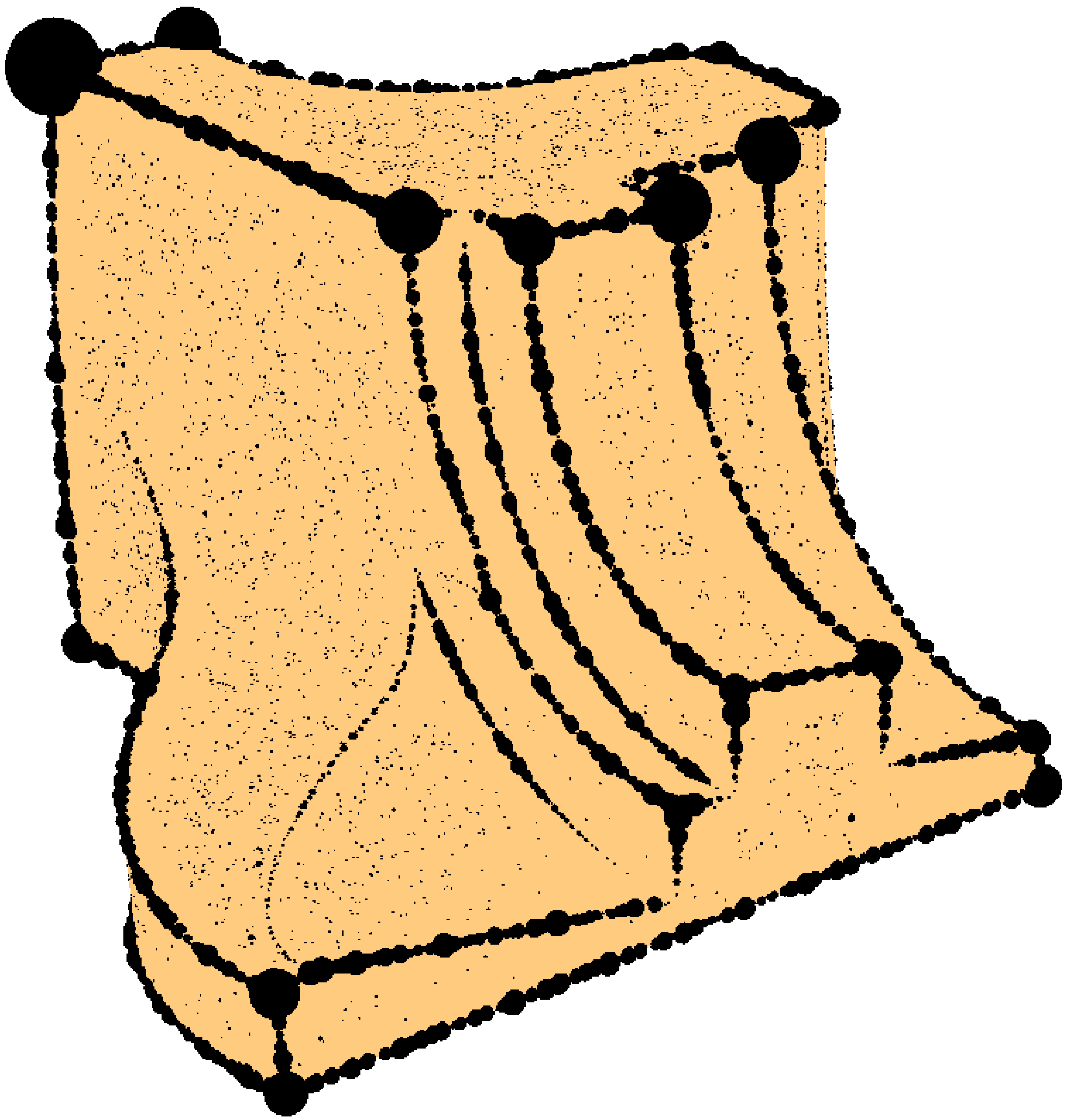

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(1) Sample points $(X_i)_{1 \leq i \leq N}$ in P^r

(2) Compute the projection of each point (X_i) on P: $p_i \leftarrow \mathrm{p}_P(X_i)$.

(3) Consider $\mu_N = \frac{1}{N} \sum_i \delta_{p_i}$

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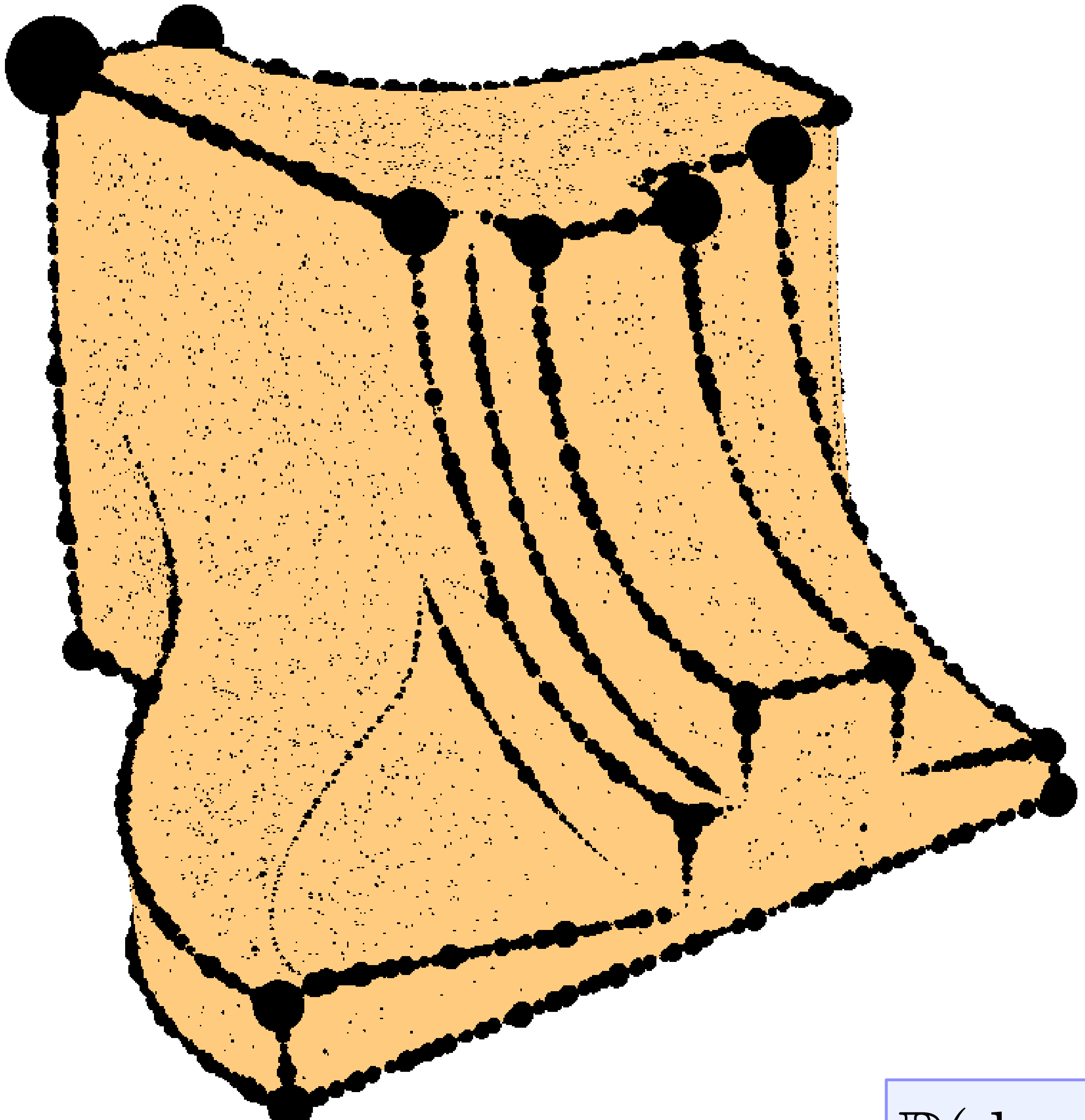

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$$\mathbb{P}(\mathrm{d_{bL}}(\mu_N, \mu) \geq \varepsilon) \leq 2 \exp(|P| \ln(16/\varepsilon) - N\varepsilon^2)$$

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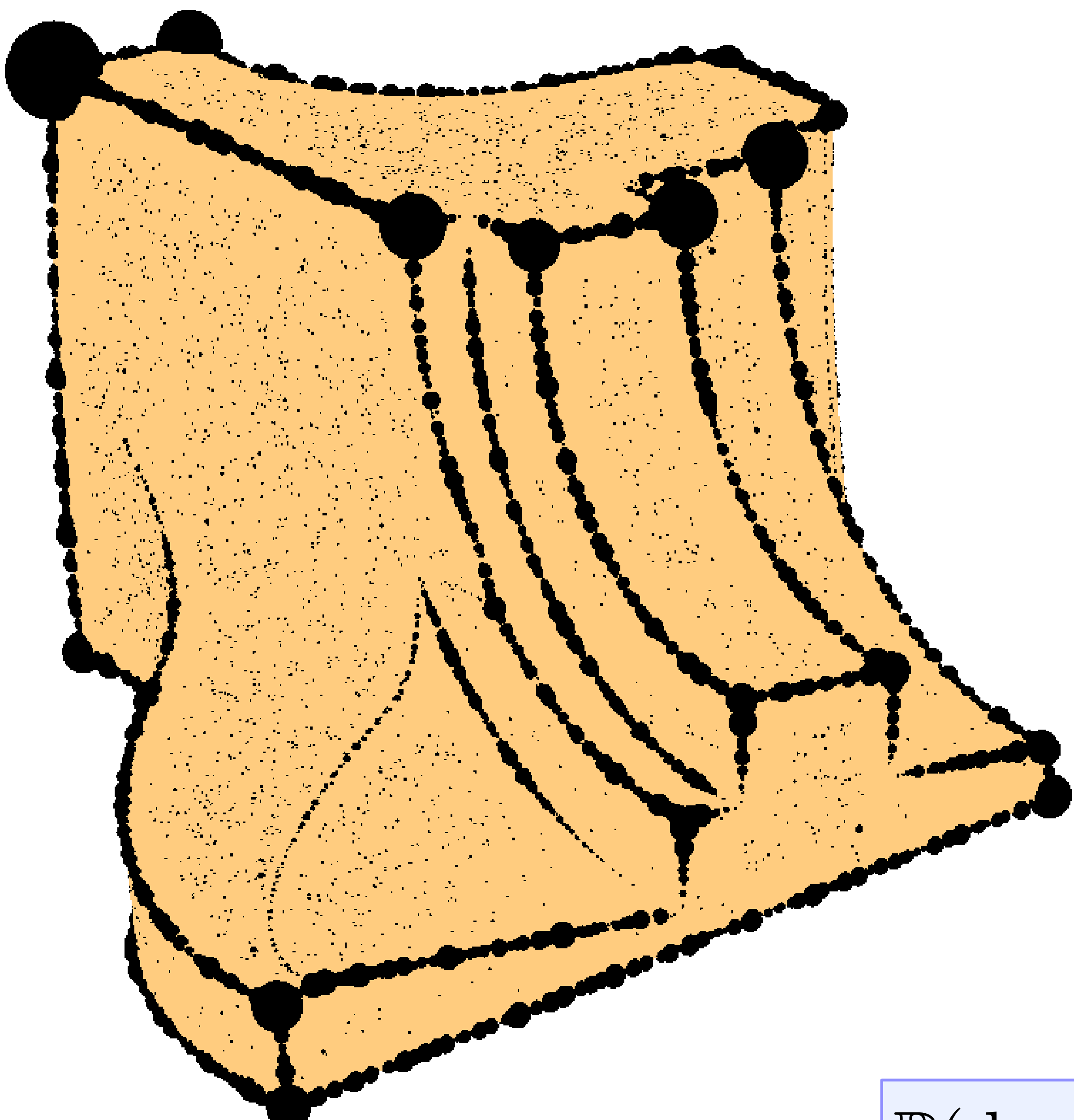
Computation of boundary measures and VCMs

Boundary measures of a finite sets can be computed via **Monte-Carlo**:



Input: $P \subseteq \mathbb{R}^d$, $r > 0$, $N \in \mathbb{N}$

Output: $\mu_N \sim \mu := \frac{\mu_{P,P^r}}{\mathrm{vol}^d(P^r)}$

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(3) Consider $\mu_N = \frac{1}{N} \sum_i \delta_{p_i}$

$\nu_N = \frac{1}{N} \sum_i (X_i - p_i) \otimes (X_i - p_i) \delta_{p_i}$

$$\mathbb{P}(\mathrm{d_{bL}}(\mu_N, \mu) \geq \varepsilon) \leq 2 \exp(|P| \ln(16/\varepsilon) - N\varepsilon^2)$$

$P = 15k$ *points on the "fandisk"*