Tensor valuations

Rolf Schneider

Universität Freiburg

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7. Properties of the Minkowski tensors

Recall that

$$\Phi_n^{r,0}(K) = \Psi_r(K) = \frac{1}{r!} \int_K x^r \, \mathrm{d}x$$

and

$$\Phi_k^{r,s}(K) = \frac{1}{r!s!} \frac{\omega_{n-k}}{\omega_{n-k+s}} \int_{\Sigma^n} x^r u^s \, \Lambda_k(K, \mathrm{d}(x,u))$$

for $K \in \mathcal{K}^n$, $k = 1, \ldots, n-1$ and $r, s \in \mathbb{N}_0$.

Recall also the convention

$$\Phi_k^{r,s} := 0 \quad \text{if } k \notin \{0, \ldots, n\} \text{ or } r \notin \mathbb{N}_0 \text{ or } s \notin \mathbb{N}_0 \text{ or } k = n,\ s \neq 0.$$

Steiner-type formula:

$$\Psi_r(K + \rho B^n) = \sum_{k=0}^{n+r} \rho^{n+r-k} \kappa_{n+r-k} V_k^{(r)}(K),$$

where

$$V_k^{(r)} = \sum_{s \in \mathbb{N}_0} \Phi_{k-r+s}^{r-s,s}.$$

For $r = 0$, this reduces to the classical Steiner formula for the volume.

We indicate the main steps of a proof of this formula.

For this, we require an extension of spherical coordinates, with the sphere replaced by the boundary of a general convex body.

First, the support measures satisfy a Steiner-type formula:

Write $K_\rho := K + \rho B^n$ and define the mapping $\tau_\rho : \Sigma^n \to \Sigma^n$ by

$$\tau_\rho(x, u) := (x + \rho u, u).$$

Then

$$2\Lambda_{n-1}(K_\rho, \cdot) = \sum_{k=0}^{n-1} \rho^{n-k-1} \omega_{n-k} \tau_\rho \Lambda_k(K, \cdot),$$

where $\tau_\rho \Lambda_k(K, \cdot)$ is the image measure of $\Lambda_k(K, \cdot)$ under τ_ρ.

Using this, the following formula can be proved:

Lemma 7.1 *Let $K \in \mathcal{K}^n$, and let $f : \mathbb{R}^n \setminus K \to \mathbb{R}$ be a nonnegative measurable function. Then*

$$\int_{\mathbb{R}^n \setminus K} f(x)\,\mathrm{d}x$$

$$= \sum_{j=0}^{n-1} \omega_{n-j} \int_0^\infty t^{n-j-1} \int_{\Sigma^n} f(x+tu)\,\Lambda_j(K, \mathrm{d}(x,u))\,\mathrm{d}t.$$

This is used to compute the last term of

$$\Psi_r(K + \rho B^n) = \Psi_r(K) + \frac{1}{r!} \int_{K_\rho \setminus K} x^r \,\mathrm{d}x.$$

$$\begin{aligned}
&\int_{K_\rho\setminus K} x^r \,\mathrm{d}x \\
&= \sum_{j=0}^{n-1} \omega_{n-j} \int_0^\rho t^{n-j-1} \int_{\Sigma^n} (x+tu)^r \,\Lambda_j(K,\mathrm{d}(x,u))\,\mathrm{d}t \\
&= \sum_{j=0}^{n-1} \omega_{n-j} \int_0^\rho t^{n-j-1} \int_{\Sigma^n} \sum_{s=0}^{r} \binom{r}{s} x^{r-s}u^s t^s \,\Lambda_j(K,\mathrm{d}(x,u))\,\mathrm{d}t \\
&= \sum_{j=0}^{n-1} \sum_{s=0}^{r} \omega_{n-j} \binom{r}{s} \frac{\rho^{n-j+s}}{n-j+s} \int_{\Sigma^n} x^{r-s}u^s \,\Lambda_j(K,\mathrm{d}(x,u)).
\end{aligned}$$

Introducing the index $k = j + r - s$ and using the definition of $\Phi_k^{r,s}$, we obtain the assertion.

For the next goals, recall Hadwiger's characterization theorem.

It determines the real vector space of all mappings $\varphi : \mathcal{K}^n \to \mathbb{R}$ which are

- valuations,
- rigid motion invariant,
- continuous.

The result is that this vector space is spanned by the intrinsic volumes $V_0, \dots, V_n$.

$V_0, \dots, V_n$ are linearly independent, because they have different degrees of homogeneity.

Hence, the vector space in question has dimension $n + 1$.

Extensions to higher ranks?

Rank 1 is an old result (Hadwiger and R.S. 1971):

Theorem 7.1 *The real vector space of all mappings* $\psi : \mathcal{K}^n \to \mathbb{R}^n$ *which are*

- *valuations,*
- *rotation equivariant, and such that* $\psi(K+t) - \psi(K)$ *is always parallel to* t*,*
- *continuous,*

is spanned by the mappings

$$K \mapsto \int_K x \, C_j(K, \mathrm{d}x), \qquad j = 0, \dots, n,$$

(the moment vectors of the curvature measures).

Again, the mappings

$$K \mapsto \int_K x\, C_j(K, \mathrm{d}x), \qquad j = 0, \dots, n,$$

have different degrees of homogeneity and hence are linearly independent.

The dimension is again $n+1$, since the term of degree $n+1$ in the Steiner formula for $\Phi_1^{n,0}$ is

$$\int_{B^n} x\, \mathrm{d}x = 0.$$

The cases of higher ranks are more complicated.

Each Minkowski tensor $\Phi_k^{r,s}$ defines a mapping

$$\Gamma : \mathcal{K}^n \to \mathbb{T}^p, \quad p = r + s,$$

which is

- a valuation,
- continuous,
- isometry covariant, i.e., it is rotation covariant,

$$\Gamma(\vartheta K) = \vartheta \Gamma(K) \quad \text{for } K \in \mathcal{K}^n \text{ and } \vartheta \in O(n),$$

and has polynomial translation behaviour,

$$\Gamma(K + t) = \sum_{j=0}^{p} \Gamma_{p-j}(K) t^j \quad \text{for } K \in \mathcal{K}^n \text{ and } t \in \mathbb{R}^n,$$

with tensors $\Gamma_{p-j}(K) \in \mathbb{T}^{p-j}$, independent of t.

A new aspect for ranks ≥ 2: There is constant mapping

$$\Gamma : \mathcal{K}^n \to \mathbb{T}^2$$

that has all the listed properties, namely the metric tensor Q,

$$Q(a, b) := \langle a, b \rangle \quad \text{for } a, b \in \mathbb{R}^n.$$

The constant mapping $\Gamma = Q$ is trivially a valuation, continuous, and has polynomial behaviour under translations. Since

$$Q(a, b) = \langle a, b \rangle = \langle \vartheta^{-1} a, \vartheta^{-1} b \rangle = (\vartheta Q)(a, b),$$

it is also rotation covariant.

It follows that the mappings

$$K \mapsto Q^m \Phi_k^{r,s}(K)$$

have the same properties.

Theorem 7.2 (Alesker 1999) *Let $p \in \mathbb{N}_0$. The real vector space of all mappings $\Gamma : \mathcal{K}^n \to \mathbb{T}^p$ which are*

- *valuations,*
- *isometry covariant,*
- *continuous,*

is spanned by the basic tensor valuations

$$Q^m \Phi_k^{r,s},$$

where $m, r, s \in \mathbb{N}_0$ satisfy $2m + r + s = p$ and where $k \in \{0, \dots, n\}$, but $s = 0$ if $k = n$.

For $p \geq 2$, the basic tensor valuations are not linearly independent:

They satisfy the McMullen relations.

The McMullen relations

The crucial relation is the identity

$$Q\Phi_n^{r-1,0} = 2\pi\Phi_{n-1}^{r,1}.$$

Explicitly, this reads

$$Q\frac{1}{(r-1)!}\int_K x^{r-1}\,\mathrm{d}x = \frac{2}{r!}\int_{\Sigma^n} x^r u\,\Lambda_{n-1}(K,\mathrm{d}(x,u)).$$

For smooth K (which is sufficient to consider), this reads

$$Q\frac{1}{(r-1)!}\int_K x^{r-1}\,\mathrm{d}x = \frac{1}{r!}\int_{\mathrm{bd}\,K} x^r u(K,x)\,\mathcal{H}^{n-1}(\mathrm{d}x),$$

where $u(K,x)$ is the unique outer unit normal vector of K at its boundary point x.

This invites application of the divergence theorem.

$x_1, \ldots, x_n$ coordinates of $x \in \mathbb{R}^n$ w.r.t. ON basis $(e_1, \ldots, e_n)$

For given $i_1, \ldots, i_r, j \in \{1, \ldots, n\}$, define a vector field v by

$$v(x) := x_{i_1} \cdots x_{i_r} e_j.$$

To this and the convex body K, apply the divergence theorem

$$\int_K \operatorname{div} v(x) \, \mathrm{d}x = \int_{\operatorname{bd} K} \langle v(x), u(K, x) \rangle \, \mathcal{H}^{n-1}(\mathrm{d}x).$$

This yields

$$\int_K \sum_{k=1}^{r} \delta_{i_k j} \, x_{i_1} \cdots \check{x}_{i_k} \cdots x_{i_r} \, \mathrm{d}x = \int_{\operatorname{bd} K} x_{i_1} \cdots x_{i_r} \langle e_j, u(K, x) \rangle \, \mathcal{H}^{n-1}(\mathrm{d}x).$$

(δ is the Kronecker symbol, and $\check{x}_m$ indicates that x_m is deleted.)

Using tensor notation, the obtained equation

$$\int_K \sum_{k=1}^{r} \delta_{i_k j}\, x_{i_1} \cdots \check{x}_{i_k} \cdots x_{i_r}\, \mathrm{d}x = \int_{\mathrm{bd}\, K} x_{i_1} \cdots x_{i_r} \langle e_j, u(K,x)\rangle\, \mathcal{H}^{n-1}(\mathrm{d}x)$$

can be written as

$$\sum_{k=1}^{r} Q(e_{i_k}, e_j)\Psi_{r-1}(K)(e_{i_1}, \ldots, \check{e}_{i_k}, \ldots, e_{i_r})$$

$$(*)$$

$$= \frac{1}{(r-1)!} \int_{\mathrm{bd}\, K} x^r(e_{i_1}, \ldots, e_{i_r}) u(K,x)(e_j)\, \mathcal{H}^{n-1}(\mathrm{d}x).$$

It remains to see that this is the identity we want.

What we want is

$$Q\Psi_{r-1}(K) = \frac{1}{r!} \int_{\mathrm{bd}\, K} x^r u(K,x)\, \mathcal{H}^{n-1}(\mathrm{d}x). \tag{1}$$

We need to check this only on $(r+1)$-tuples $(e_{i_1}, \ldots, e_{i_{r+1}})$ of basis vectors.

Left side of (1): By the definition of the symmetric tensor product,

$$(r+1)!(Q\Psi_{r-1}(K))(e_{i_1}, \ldots, e_{i_{r+1}})$$

$$= \sum_{\sigma \in \mathcal{S}(r+1)} Q(e_{i_{\sigma(1)}}, e_{i_{\sigma(2)}}) \Psi_{r-1}(K)(e_{i_{\sigma(3)}}, \ldots, e_{i_{\sigma(r+1)}}). \tag{2}$$

Right side of (1):

$$(r+1)!\,\frac{1}{r!}\int_{\mathrm{bd}\,K}(x^r u(K,x))(e_{i_1},\dots,e_{i_{r+1}})\,\mathcal{H}^{n-1}(\mathrm{d}x)$$

$$=\frac{1}{r!}\sum_{\sigma\in\mathcal{S}(r+1)}\int_{\mathrm{bd}\,K}x^r(e_{i_{\sigma(1)}},\dots,e_{i_{\sigma(r)}})u(K,x)(e_{i_{\sigma(r+1)}})\,\mathcal{H}^{n-1}(\mathrm{d}x)$$

using (*)

$$=\frac{1}{r}\sum_{k=1}^{r}\sum_{\sigma\in\mathcal{S}(r+1)}Q(e_{\sigma(i_k)},e_{\sigma(i_{r+1})})$$

$$\times\Psi_{r-1}(K)(e_{\sigma(i_1)},\dots,\check{e}_{\sigma(i_k)},\dots,e_{\sigma(i_r)})$$

$$=\sum_{\sigma\in\mathcal{S}(r+1)}Q(e_{\sigma(1)},e_{\sigma(i_2)})\Psi_{r-1}(K)(e_{\sigma(3)},\dots,e_{\sigma(i_{r+1})}).$$

This is the same as in (2). □

From the obtained relation

$$Q\Phi_n^{r-1,0} = 2\pi\Phi_{n-1}^{r,1},$$

further identities can be derived by applying it to parallel bodies.

For this, we use strictly convex K and write the identity in the form

$$Q\Psi_{r-1}(K) = \frac{1}{r!}\int_{\mathbb{S}^{n-1}} x(K,u)^r u\, S_{n-1}(K,\mathrm{d}u),$$

where $x(K,u)$ is the unique boundary point of K at which u is attained as outer normal vector.

We apply this to $K+\rho B^n$.

For the left side we get, using the Steiner formula ,

$$Q\Psi_{r-1}(K+\rho B^n) = \sum_{k=0}^{n+r-1} \rho^{n+r-1-k}\kappa_{n+r-1-k} QV_k^{(r-1)}(K).$$

For the right side, we use

$$x(K + \rho B^n, u) = x(K, u) + \rho u$$

and the Steiner-type formula

$$S_{n-1}(K + \rho B^n, \cdot) = \sum_{i=0}^{n-1} \rho^{n-1-i} \binom{n-1}{i} S_i(K, \cdot).$$

Inserting this and rearranging, we get

Theorem 7.3 (McMullen 1997) *For $r \in \mathbb{N}$ with $r \geq 2$ and $k \in \{0, \ldots, n + r - 2\}$,*

$$Q \sum_{s \in \mathbb{N}_0} \Phi_{k-r+s}^{r-s,s-2} = 2\pi \sum_{s \in \mathbb{N}_0} s \Phi_{k-r+s}^{r-s,s}.$$

For rank one, the McMullen relations also hold, but only express the well-known fact that

$$\int_{\mathbb{S}^{n-1}} u\, S_j(K, \mathrm{d}u) = 0$$

for $j = 0, \ldots, n-1$.

For rank two, the McMullen relations are given by

$$Q\Phi_k^{0,0} = 2\pi\Phi_{k-1}^{1,1} + 4\pi\Phi_k^{0,2}, \qquad k = 0, \ldots, n.$$

We recall that

$$\begin{aligned}
\Phi_k^{0,0}(K) &= V_k, \\
\Phi_{k-1}^{1,1}(K) &= a_k \int_{\Sigma^n} xu\, \Lambda_{k-1}(K, \mathrm{d}(x,u)) \quad \text{for } k \geq 1, \quad \Phi_{-1}^{1,1}(K) = 0, \\
\Phi_k^{0,2}(K) &= b_k \int_{\Sigma^n} u^2\, \Lambda_k(K, \mathrm{d}(x,u)) \quad \text{for } k \leq n-1, \quad \Phi_n^{0,2}(K) = 0,
\end{aligned}$$

with positive constants a_k, b_k.

The following was proved by Hug, R.S. and R. Schuster (2008).

Theorem 7.4 *Any nontrivial linear relation between basic tensor valuations* $Q^m\Phi_k^{r,s}$ *can be obtained by multiplying suitable McMullen relations by powers of* Q *and by taking linear combinations of relations obtained in this way.*

This opened the way to the determination of dimensions and bases.

Let $T_{p,k}$ denote the real vector space of all mappings $\mathcal{K}^n \to \mathbb{T}^p$ that are continuous, isometry covariant valuations and homogeneous of degree k.

dim $T_{p,k}$ has been determined *loc. cit.*

Example for explicit bases in rank two:

- $T_{2,0}$: a basis is $\{Q\Phi_0^{0,0}\}$.
- $T_{2,1}$: a basis is $\{\Phi_1^{0,2}, Q\Phi_1^{0,0}\}$.
- $T_{2,k}$ for $k = 2, \ldots, n-1$: a basis is $\{\Phi_k^{0,2}, \Phi_{k-2}^{2,0}, Q\Phi_k^{0,0}\}$.
- $T_{2,n}$: a basis is $\{\Phi_{n-2}^{2,0}, Q\Phi_n^{0,0}\}$.
- $T_{2,k}$ for $k = n+1, n+2$: a basis is $\{\Phi_{k-2}^{2,0}\}$.

Thus, the vector space of continuous, isometry covariant tensor valuations of rank two has dimension $3n+1$.

The Minkowski tensors have measure-valued extensions.

We abbreviate

$$c_{n,k}^{r,s} := \frac{1}{r!s!}\,\frac{\omega_{n-k}}{\omega_{n-k+s}}$$

and define the local Minkowski tensors

$$\phi_k^{r,s} : \mathcal{K}^n \times \mathcal{B}(\Sigma^n) \to \mathbb{T}^{r+s}$$

by

$$\phi_k^{r,s}(K,\eta) := c_{n,k}^{r,s} \int_\eta x^r u^s \,\Lambda_k(K, \mathrm{d}(x,u))$$

for $\eta \in \mathcal{B}(\Sigma^n)$, $r, s \in \mathbb{N}_0$, $k \in \{0, \ldots, n-1\}$.

For $\eta \in \mathcal{B}(\Sigma^n)$, $t \in \mathbb{R}^n$ and $\vartheta \in \mathrm{O}(n)$, write

$$\eta + t := \{(x+t, u) : (x, u) \in \eta\}$$
$$\vartheta\eta := \{(\vartheta x, \vartheta u) : (x, u) \in \eta\}$$

The following properties of a mapping $\Gamma : \mathcal{K}^n \times \mathcal{B}(\Sigma^n) \to \mathbb{T}^p$ will be important.

- Γ has polynomial translation behaviour of degree q, where $0 \leq q \leq p$, if

$$\Gamma(K+t, \eta+t) = \sum_{j=0}^{q} \frac{1}{j!} \Gamma_{p-j}(K, \eta) t^j$$

with tensors $\Gamma_{p-j}(K, \eta) \in \mathbb{T}^{p-j}$, for $K \in \mathcal{K}^n$, $\eta \in \mathcal{B}(\Sigma^n)$, $t \in \mathbb{R}^n$.

- Γ is rotation covariant if $\Gamma(\vartheta K, \vartheta\eta) = \vartheta\Gamma(K, \eta)$ for $K \in \mathcal{K}^n$, $\eta \in \mathcal{B}(\Sigma^n)$, $\vartheta \in \mathrm{O}(n)$.

- Γ is isometry covariant (of degree q) if it has polynomial translation behaviour of some degree $q \leq p$ (and hence of degree p) and is rotation covariant.

- Γ is locally defined if for $\eta \in \mathcal{B}(\Sigma^n)$ and $K, K' \in \mathcal{K}^n$ with $\eta \cap \mathrm{Nor}\, K = \eta \cap \mathrm{Nor}\, K'$ the equality $\Gamma(K, \eta) = \Gamma(K', \eta)$ holds.

- If $\Gamma(K, \cdot)$ is a $\mathbb{T}^p$-valued measure for each $K \in \mathcal{K}^n$, then Γ is weakly continuous if for each sequence $(K_i)_{i \in \mathbb{N}}$ of convex bodies in $\mathcal{K}^n$ converging to a convex body K the relation

$$\lim_{i \to \infty} \int_{\Sigma^n} f \,\mathrm{d}\Gamma(K_i, \cdot) = \int_{\Sigma^n} f \,\mathrm{d}\Gamma(K, \cdot)$$

holds for all continuous functions $f : \Sigma^n \to \mathbb{R}$.

In the previous definitions, the set $\mathcal{K}^n$ may be replaced by $\mathcal{P}^n$.

The local Minkowski tensor

$$\Gamma = \phi_k^{r,s}$$

has the following properties (they follow from properties of the support measures):

- For each $K \in \mathcal{K}^n$, $\Gamma(K, \cdot)$ is a $\mathbb{T}^{r+s}$-valued measure,
- Γ is weakly continuous,
- For each $\eta \in \mathcal{B}(\Sigma^n)$, $\Gamma(\cdot, \eta)$ is measurable,
- For each $\eta \in \mathcal{B}(\Sigma^n)$, $\Gamma(\cdot, \eta)$ is a valuation,
- The mapping Γ is isometry covariant,
- The mapping Γ is locally defined.

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- The mapping Γ is isometry covariant,
- The mapping Γ is locally defined.

Main goal: to determine all mappings with these properties

Local tensor valuations on polytopes

$P \in \mathcal{P}^n$ a polytope

$\mathcal{F}_k(P)$ set of its k-dimensional faces

$\nu(P,F)$ set of unit normal vectors of P at F

The local Minkowski tensors of a polytope P have the explicit representation

$$\phi_k^{r,s}(P,\eta) = C_{n,k}^{r,s} \sum_{F \in \mathcal{F}_k(P)} \int_F \int_{\nu(P,F)} \mathbf{1}_\eta(x,u) x^r u^s \, \mathcal{H}^{n-k-1}(\mathrm{d}u) \, \mathcal{H}^k(\mathrm{d}x),$$

where

$$C_{n,k}^{r,s} := (r!s!\omega_{n-k+s})^{-1}$$

and $\mathbf{1}_\eta$ is the characteristic function of η.

This representation can be modified, retaining all the listed properties, with the possible exception of weak continuity.

$L \subset \mathbb{R}^n$ linear subspace

$\pi_L : \mathbb{R}^n \to L$ orthogonal projection

$$Q_L(a,b) := \langle \pi_L a, \pi_L b \rangle \qquad \text{for } a, b \in \mathbb{R}^n.$$

Note that $Q_{\vartheta L} = \vartheta Q_L$ for $\vartheta \in O(n)$.

For a face F of P, the direction space of F is the linear subspace parallel to $\operatorname{aff} F$.

The generalized local Minkowski tensors on polytopes are defined by

$$\phi_k^{r,s,j}(P,\eta)$$

$$:= C_{n,k}^{r,s} \sum_{F\in\mathcal{F}_k(P)} Q_{L(F)}^{j} \int_F \int_{\nu(P,F)} \mathbf{1}_\eta(x,u) x^r u^s \, \mathcal{H}^{n-k-1}(\mathrm{d}u) \, \mathcal{H}^k(\mathrm{d}x),$$

for $r, s, j, k \in \mathbb{N}_0$ with $1 \le k \le n-1$. Further, $\phi_0^{r,s,0} := \phi_0^{r,s}$.

Properties of $\Gamma = \phi_k^{r,s,j}$:

- $\Gamma(\cdot,\eta)$ is a valuation
- $\Gamma(P,\cdot)$ is a $\mathbb{T}^p$-valued measure
- isometry covariant
- locally defined

Theorem 8.1 *For $p \in \mathbb{N}_0$, let $T_p(\mathcal{P}^n)$ denote the real vector space of all mappings*

$$\Gamma : \mathcal{P}^n \times \mathcal{B}(\Sigma^n) \to \mathbb{T}^p$$

with the following properties.

(a) $\Gamma(P, \cdot)$ *is a $\mathbb{T}^p$-valued measure, for each $P \in \mathcal{P}^n$;*

(b) Γ *is isometry covariant;*

(c) Γ *is locally defined.*

Then a basis of $T_p(\mathcal{P}^n)$ is given by the mappings

$$Q^m \phi_k^{r,s,j},$$

where $m, r, s, j \in \mathbb{N}_0$ satisfy $2m + 2j + r + s = p$ and where $k \in \{0, \dots, n-1\}$, but $j = 0$ if $k \in \{0, n-1\}$.

Most of the theorem (slightly stronger assumptions, without linear independence) was proved by R.S. 2013, the rest by Hug and R.S. 2014.

By an induction argument, stepwise reducing the degree of the polynomial translation behaviour, the proof of Theorem 8.1 can be reduced to the translation invariant case.

Most of the theorem (slightly stronger assumptions, without linear independence) was proved by R.S. 2013, the rest by Hug and R.S. 2014.

By an induction argument, stepwise reducing the degree of the polynomial translation behaviour, the proof of Theorem 8.1 can be reduced to the translation invariant case.

Theorem 8.2 *Let $p \in \mathbb{N}_0$. Let $\Gamma : \mathcal{P}^n \times \mathcal{B}(\Sigma^n) \to \mathbb{T}^p$ be a mapping with the following properties.*

(a) $\Gamma(P, \cdot)$ *is a $\mathbb{T}^p$-valued measure, for each $P \in \mathcal{P}^n$;*

(b) Γ *is translation invariant and rotation covariant;*

(c) Γ *is locally defined.*

Then Γ is a linear combination, with constant coefficients, of the mappings $Q^m \phi_k^{0,s,j}$, where $m, s, j \in \mathbb{N}_0$ satisfy $2m + 2j + s = p$ and where $k \in \{0, \dots, n-1\}$, but $j = 0$ if $k \in \{0, n-1\}$.

Some elements of the proof of Theorem 8.2

To prove an equality for measures on $\mathcal{B}(\Sigma^n)$, it is sufficient to prove equality on product sets $\beta \times \omega$, with $\beta \in \mathcal{B}(\mathbb{R}^n)$ and $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$. In this case

$$\Gamma(P, \beta \times \omega) = \sum_{k=0}^{n-1} \sum_{F \in \mathcal{F}_k(P)} \Gamma(P, (\beta \cap \operatorname{relint} F) \times (\omega \cap \nu(P, F))).$$

Therefore, we have to determine $\Gamma(F, \beta \times \omega)$ if

- $k \in \{0, \dots, n-1\}$
- $L \subset \mathbb{R}^n$ is a k-dimensional linear subspace
- $\beta \subset L$ is a bounded Borel set
- $\omega \subset \mathbb{S}^{n-1} \cap L^\perp$ is a Borel set
- $F \subset L$ is a k-dimensional polytope with $\beta \subset \operatorname{relint} F$

A standard characterization of Lebesgue measure yields

$$\Gamma(F, \beta \times \omega) = a(L, \omega)\mathcal{H}^k(\beta).$$

The crucial step is to show that the tensorial factor is of the form

$$a(L, \omega) = \sum_{j=0}^{\lfloor p/2 \rfloor} Q_L^j \sum_{i=0}^{\lfloor p/2 \rfloor} c_{pkij} Q_{L^\perp}^i \int_\omega u^{p-2j-2i} \, \mathcal{H}^{n-k-1}(\mathrm{d}u)$$

with real constants c_{pkij}.

For this, two general lemmas are used.

Lemma 8.1 *Let $L \subset \mathbb{R}^n$ be a linear subspace. Let $r \in \mathbb{N}_0$, let $T \in \mathbb{T}^r$ be a tensor satisfying $\vartheta T = T$ for each $\vartheta \in \mathrm{SO}(n)$ that fixes $L^\perp$ pointwise. Then*

$$T = \sum_{j=0}^{\lfloor r/2 \rfloor} Q_L^j \pi_{L^\perp}^* T^{(r-2j)}$$

with tensors $T^{(r-2j)} \in \mathbb{T}^{r-2j}(L^\perp)$, $j = 0, \dots, \lfloor r/2 \rfloor$.

Here

$$(\pi_{L^\perp}^* T)(x_1, \dots, x_p) := T(\pi_{L^\perp} x_1, \dots, \pi_{L^\perp} x_p) \quad \text{for } x_1, \dots, x_p \in \mathbb{R}^n.$$

The proof of Lemma 8.1 is based on the fact that the algebra of symmetric tensors on $\mathbb{R}^n$ is isomorphic to the polynomial algebra on $\mathbb{R}^n$, and it uses some manipulations with polynomials.

Lemma 8.2 *Let $r \in \mathbb{N}_0$, and let $\mu : \mathcal{B}(\mathbb{S}^{n-1}) \to \mathbb{T}^r$ be a $\mathbb{T}^r$-valued measure satisfying*

$$\mu(\vartheta\omega) = (\vartheta\mu)(\omega) \quad \textit{for } \omega \in \mathcal{B}(\mathbb{S}^{n-1}) \textit{ and } \vartheta \in \mathrm{O}(n).$$

Then

$$\mu(\omega) = \sum_{j=0}^{\lfloor r/2 \rfloor} a_j Q^j \int_\omega u^{r-2j}\, \mathcal{H}^{n-1}(\mathrm{d}u), \qquad \omega \in \mathcal{B}(\mathbb{S}^{n-1}),$$

with real constants a_j, $j = 0, \dots, \lfloor r/2 \rfloor$.

The proof of Lemma 8.2 is based on a characterization of spherical Lebesgue measure, the Radon–Nikodym theorem, Lemma 8.1, invariance properties, and Lebesgue's differentiation theorem.