Valuations on Lattice Polytopes
Part I

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Tensor Valuations at Sandbjerg, September 2014

Valuations on Convex Bodies

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- A function $\mathrm{z} : \mathcal{F} \to \langle \mathbb{A}, + \rangle$ is a *valuation* $\iff$

$$\mathrm{z}(K) + \mathrm{z}(L) = \mathrm{z}(K \cup L) + \mathrm{z}(K \cap L)$$

for all $K, L \in \mathcal{F}$ such that $K \cup L, K \cap L \in \mathcal{F}$.

The Hadwiger Classification Theorem 1952

Theorem

A functional $\mathrm{z}: \mathcal{K}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous and rigid motion invariant valuation*

$$\Longleftrightarrow$$

$\exists\ c_0, c_1, \ldots, c_n \in \mathbb{R}$ *such that*

$$\mathrm{z}(K) = c_0\, V_0(K) + \cdots + c_n\, V_n(K)$$

for every $K \in \mathcal{K}(\mathbb{R}^n)$.

$V_0(K), \ldots, V_n(K)$	intrinsic volumes of K
V_n	n-dimensional volume
$2\, V_{n-1}(K) = S(K)$	surface area

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New proof by Dan Klain 1995

Classification of Valuations on Convex Bodies

- **Real valuations**:
  Blaschke 1937, Hadwiger 1949, McMullen 1977, Klain 1995, Alesker 1998, Ludwig 1999, Ludwig & Reitzner 1999, Schuster 2006, Bernig & Fu 2011, Haberl & Parapatits 2014, ...
- **Vector and tensor valuations**:
  Hadwiger & Schneider 1971, Schneider 1972, McMullen 1977, Alesker 1999, Ludwig 2002, Haberl & Parapatits 2014+, ...
- **Convex body valued and star body valued valuations:**
  Schneider 1974 (Minkowski endomorphisms), Ludwig 2002 (Minkowski valuations), Kiderlen 2006, Haberl & Ludwig 2006, Ludwig 2006, Schneider & Schuster 2006, Schuster 2007, Haberl 2009, Abardia & Bernig 2011, Wannerer 2011, Schuster & Wannerer 2012, Abardia 2012, Parapatits 2014, Li & Yuan & Leng 2014+, ...

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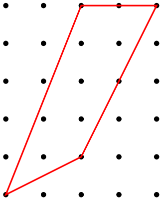

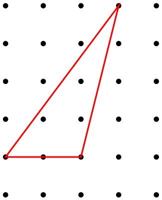

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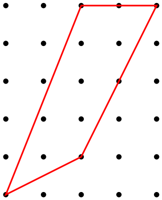

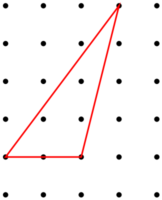

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- Applications
  - Geometry of numbers
  - Crystallography
  - Statistical physics
  - Integer programming

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for all $P, Q \in \mathcal{P}(\mathbb{Z}^n)$ such that $P \cup Q \in \mathcal{P}(\mathbb{Z}^n)$.

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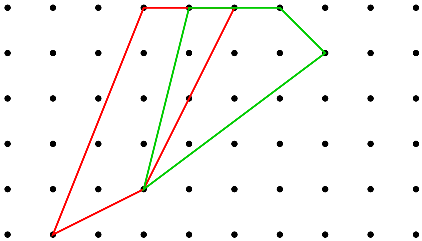

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Question.

Classification of $\mathrm{SL_n}(\mathbb{Z})$ and translation invariant valuations $\mathrm{z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}$

The Betke & Kneser Classification Theorem 1985

Theorem (Betke & Kneser: Crelle 1985)

A functional $\mathrm{z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}$ *is an* $\mathrm{SL_n}(\mathbb{Z})$ *and translation invariant valuation*

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Betke & Kneser: unimodular valuations

Classification Theorems

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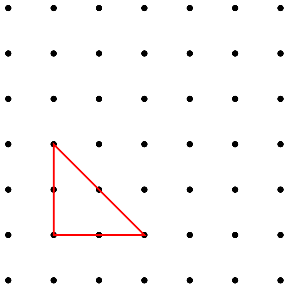

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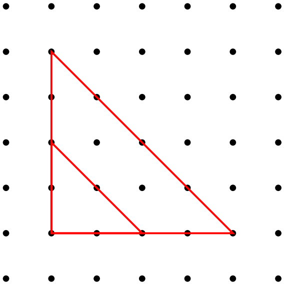

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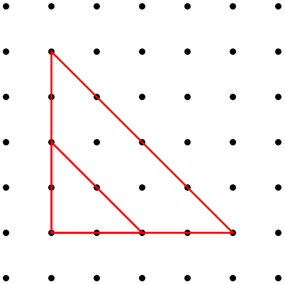

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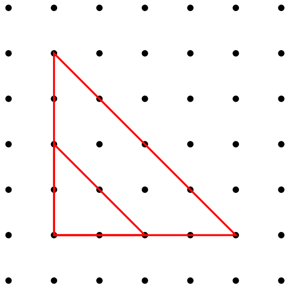

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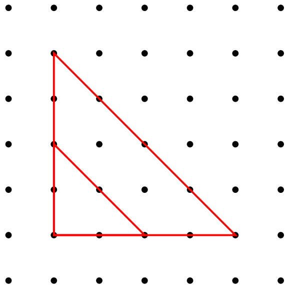

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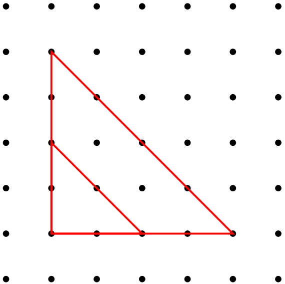

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- Eugène Ehrhart 1962

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Theorem (McMullen 1975)

If $P_1, \dots, P_m \in \mathcal{P}(\mathbb{Z}^n)$ and $k_1, \dots, k_m \in \mathbb{N}$, then $L(k_1 P_1 + \dots + k_m P_m)$ is a polynomial in $k_1, \dots, k_m$.

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Corollary (Raman Sanyal 2014)

The functional $L_1 : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}$ is Minkowski additive.

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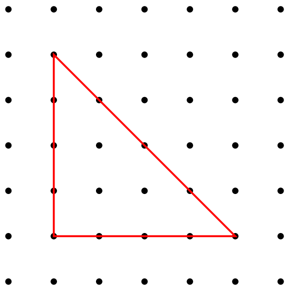

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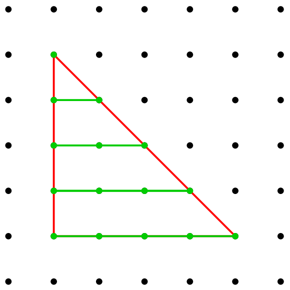

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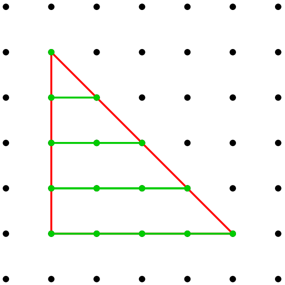

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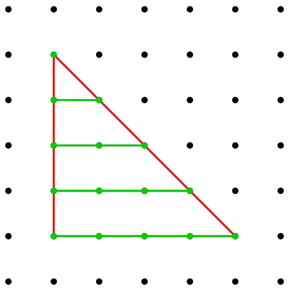

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- S basic n-dimensional simplex $\Rightarrow$

  $S = \phi T_n$ with $\phi \in \mathrm{SL_n}(\mathbb{Z})$

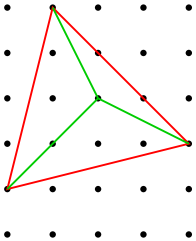
  $L(k\, S) = \sum_{i=0}^{n} L_i(S)\, k^i$

Inclusion-Exclusion Principle

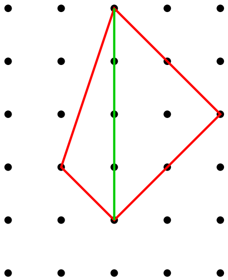
Theorem (Betke 1979; McMullen: AiM 2009)

Let $\mathrm{z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{A}$ *be a valuation, where* $\mathbb{A}$ *is an Abelian group. If the lattice polytopes* $P_1, \dots, P_k$ *satisfy that their union and the intersection of at most* n *of them are all lattice polytopes, then*

$$\mathrm{z}(P_1 \cup \cdots \cup P_k) = \sum_{\substack{1 \leqslant i_1 < \cdots < i_m \leqslant k \\ 1 \leqslant m \leqslant \min\{k,n\}}} (-1)^{m-1}\, \mathrm{z}(P_{i_1} \cap \cdots \cap P_{i_m}).$$



Dissections of Lattice Polytopes



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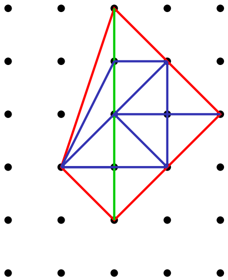

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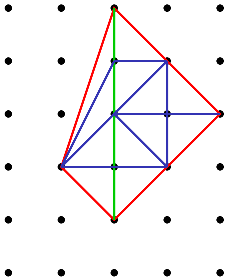

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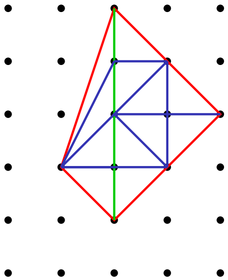

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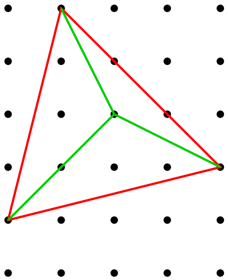
  Reeve tetrahedron [(0,0,0), (1,0,0), (0,1,0), (1,1,k)] with $k \in \mathbb{N}$

Dissections and Complementations



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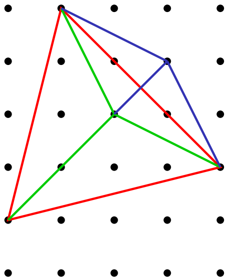

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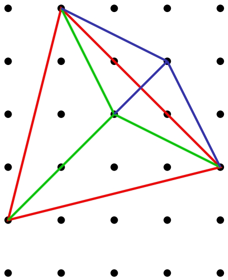

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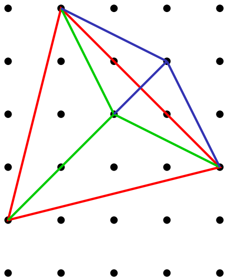

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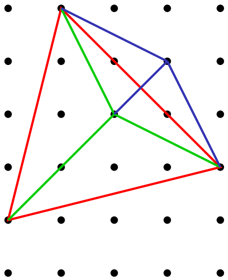
- Similar arguments for dissecting and complementing by basic simplices

  $\Rightarrow$ existence of Ehrhart polynomial

Dissections and Complementations



- Proof of the inclusion-exclusion principle
- Similar arguments for dissecting and complementing by basic simplices

  $\Rightarrow$ existence of Ehrhart polynomial

  $\Rightarrow$ Betke & Kneser theorem

Valuations on Lattice Polytopes

Theorem (Betke & Kneser 1985)

- *Every* $\mathrm{SL_n}(\mathbb{Z})$ *and translation invariant valuation* $\mathrm{z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{A}$ *is uniquely determined by its values on* $T_0, \dots, T_n$.
- *Every choice of values on* $T_0, \dots, T_n$ *in* $\mathbb{A}$ *defines an* $\mathrm{SL_n}(\mathbb{Z})$ *and translation invariant valuation.*

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- $\Rightarrow$ existence of Ehrhart polynomial
- Tensor valuations and convex body valued valuations (Part 2)

Thank you!