# Local Tensor Valuations

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(based on joint work with Rolf Schneider)

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# Getting started

We consider maps

$$\Gamma : \mathcal{P}^n \times \mathcal{B}(\mathbb{R}^n \times \mathbb{S}^{n-1}) \to \mathbb{T}^p \qquad \text{or} \qquad \Gamma : \mathcal{K}^n \times \mathcal{B}(\mathbb{R}^n \times \mathbb{S}^{n-1}) \to \mathbb{T}^p$$

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(b) isometry covariant:
- $\Gamma(K + t, \eta + t)$ is a tensor polynomial in $t \in \mathbb{R}^n$,
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(d) weakly continuous: if $\lim_{i \to \infty} K_i = K$ then

$$\lim_{i \to \infty} \int_{\mathbb{R}^n \times \mathbb{S}^{n-1}} f \,\mathrm{d}\Gamma(K_i, \cdot) = \int_{\mathbb{R}^n \times \mathbb{S}^{n-1}} f \,\mathrm{d}\Gamma(K, \cdot)$$

for all continuous functions $f : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$.

# Examples

1. For $K \in \mathcal{K}^n$ and $\eta \in \mathcal{B}(\mathbb{R}^n \times \mathbb{S}^{n-1})$

$$\phi_k^{r,s}(K,\eta) := c_{n,k}^{r,s} \int_\eta x^r u^s \, \Lambda_k(K, \mathrm{d}(x,u))$$

for $r, s, k \in \mathbb{N}_0$ with $k \leq n-1$.

Here $\Lambda_k(K,\cdot)$ is the $k$-th support measure and $x^r u^s$ is a symmetric tensor product.

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2. For $P \in \mathcal{P}^n$

$$\phi_k^{r,s,j}(P,\eta) := c_{n,k}^{r,s} \sum_{F \in \mathcal{F}_k(P)} Q_{L(F)}^j \int_F \int_{\nu(P,F)} \mathbf{1}_\eta(x,u) x^r u^s \, \mathcal{H}^{n-k-1}(\mathrm{d}u) \, \mathcal{H}^k(\mathrm{d}x)$$

   for $r, s, j, k \in \mathbb{N}_0$ with $1 \leq k \leq n-1$; $\phi_0^{r,s,0} := \phi_0^{r,s}$.

# A local characterization theorem: polytopes

For $p \in \mathbb{N}_0$, let $T_p(\mathcal{P}^n)$ denote the real vector space of all mappings $\Gamma : \mathcal{P}^n \times \mathcal{B}(\mathbb{R}^n \times \mathbb{S}^{n-1}) \to \mathbb{T}^p$ with the following properties:

(a) $\Gamma(P, \cdot)$ is a $\mathbb{T}^p$-valued measure, for each $P \in \mathcal{P}^n$;

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**Theorem 1** *A* ***basis*** *of $T_p(\mathcal{P}^n)$ is given by the mappings*

$$Q^m \phi_k^{r,s,j},$$

*where $m, r, s, j \in \mathbb{N}_0$ satisfy $2m + 2j + r + s = p$ and where $k \in \{0, \dots, n-1\}$, but $j = 0$ if $k \in \{0, n-1\}$.*

# A local characterization theorem: polytopes

For $p \in \mathbb{N}_0$, let $T_p^{(0)}(\mathcal{P}^n)$ denote the real vector space of all mappings $\Gamma : \mathcal{P}^n \times \mathcal{B}(\mathbb{R}^n \times \mathbb{S}^{n-1}) \to \mathbb{T}^p$ with the following properties:

(a) $\Gamma(P, \cdot)$ is a $\mathbb{T}^p$-valued measure, for each $P \in \mathcal{P}^n$;

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# General convex bodies

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Why did we not define $\phi_k^{r,s,j}(K,\cdot)$ for $K \in \mathcal{K}^n$ and $j \geq 1$?
At least for $k = n-1$ we can replace $Q^j_{L(F)}$ in

$$C^{r,s}_{n,n-1} \sum_{F \in \mathcal{F}_{n-1}(P)} Q^j_{L(F)} \int_F \int_{\nu(P,F)} \mathbf{1}_\eta(x,u) x^r u^s \, \mathcal{H}^0(\mathrm{d}u) \, \mathcal{H}^{n-1}(\mathrm{d}x)$$

by

$$Q^j_{L(F)} = \left(Q - u^2\right)^j = \sum_{i=0}^{j} (-1)^i \binom{j}{i} Q^{j-i} i^{2i}$$

so that

$$\phi^{r,s,j}_{n-1} = \sum_{i=0}^{j} (-1)^i \binom{j}{i} \frac{(s+2i)!\omega_{1+s+2i}}{s!\omega_{1+s}} Q^{j-i} \phi^{r,s+2i}_{n-1}.$$

# Splitting wrt degree of homogeneity

$\Gamma : \mathcal{K}^n \times \mathcal{B}(\mathbb{R}^n \times \mathbb{S}^{n-1}) \to \mathbb{T}^p$ is homogeneous of degree $k$ if

$$\Gamma(\lambda K, \lambda\eta) = \lambda^k \Gamma(K, \eta), \qquad K \in \mathcal{K}^n, \eta \in \mathcal{B}(\Sigma^n), \lambda > 0,$$

where $\lambda\eta := \{(\lambda x, u) : (x, u) \in \eta\}$.

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where $\lambda\eta := \{(\lambda x, u) : (x, u) \in \eta\}$.

**Lemma** *Let $p \in \mathbb{N}_0$. Let $\Gamma : \mathcal{K}^n \times \mathcal{B}(\Sigma^n) \to \mathbb{T}^p$ be such that*

(a) $\Gamma(K, \cdot)$ *is a $\mathbb{T}^p$-valued measure, for each $K \in \mathcal{K}^n$;*

(b) $\Gamma$ *is translation invariant and rotation covariant;*

(c) $\Gamma$ *is locally defined;*

(d) $\Gamma$ *is weakly continuous.*

*Then $\Gamma = \sum_{k=0}^{n-1} \Gamma_k$, where each $\Gamma_k$ has properties* (a) – (d) *and is homogeneous of degree $k$.*

**Proof.** The restriction $\Gamma|\mathcal{P}^n \times \mathcal{B}(\Sigma^n)$ has all the properties required to apply the characterization result for polytopes. Hence it is a linear combination of $Q^m \phi_k^{0,s,j}$ so that

$$\Gamma|\mathcal{P}^n \times \mathcal{B}(\Sigma^n) = \sum_{k=0}^{n-1} \Gamma_k$$

with $\Gamma_k : \mathcal{P}^n \times \mathcal{B}(\Sigma^n) \to \mathbb{T}^p$. This shows that

$$\Gamma(\lambda P, \lambda \eta) = \sum_{k=0}^{n-1} \lambda^k \Gamma_k(P, \eta), \qquad P \in \mathcal{P}^n, \eta \in \mathcal{B}(\Sigma^n), \lambda > 0.$$

This relation can be inverted coordinate-wise, i.e.

$$\Gamma_k(P, \eta) = \sum_{r=1}^{n} b_{kr} \Gamma(rP, r\eta), \quad k = 0, \dots, n-1,$$

and used to define $\Gamma_k$ for general convex bodies.

# Weakly continuous extension for $j = 1$

The normal bundle of $K \in \mathcal{K}^n$ is the $(n-1)$-rectifiable set

$$\operatorname{Nor} K := \{(x, u) \in \partial K \times \mathbb{S}^{n-1} : \langle x, u \rangle = h(K, u)\}.$$

For $\mathcal{H}^{n-1}$ almost all $(x, u) \in \operatorname{Nor} K$, the tangent space $\mathrm{Tan}^{n-1}(K, (x, u))$ is an $(n-1)$-dimensional linear subspace spanned by $a_1(x, u), \dots, a_{n-1}(x, u)$, where

$$a_i(x, u) := \left( \frac{1}{\sqrt{1 + k_i(x, u)^2}}\, b_i(x, u), \frac{k_i(x, u)}{\sqrt{1 + k_i(x, u)^2}}\, b_i(x, u) \right),$$

$b_1(x, u), \dots, b_{n-1}(x, u)$ is a suitably oriented ONB of $u^\perp$, and $k_i(x, u) \in [0, \infty]$.

We define an $(n-1)$-vector

$$a_K(x,u) := a_1(x,u) \wedge \ldots \wedge a_{n-1}(x,u),$$

which orients $\mathrm{Tan}^{n-1}(K,(x,u))$ and then an $(n-1)$-dimensional current in $\mathbb{R}^{2n}$ by

$$T_K := \left(\mathcal{H}^{n-1} \llcorner \operatorname{Nor} K\right) \wedge a_K,$$

the normal cycle of $K$. More explicitly,

$$T_K(\varphi) = \int_{\operatorname{Nor} K} \langle a_K(x,u), \varphi(x,u)\rangle \, \mathcal{H}^{n-1}(d(x,u)),$$

for all $\mathcal{H}^{n-1} \llcorner \operatorname{Nor} K$-integrable functions $\varphi : \mathbb{R}^{2n} \to \bigwedge^{n-1} \mathbb{R}^{2n}$.

# Lipschitz–Killing forms

Differential forms $\varphi_k : \mathbb{R}^{2n} \to \bigwedge^{n-1} \mathbb{R}^{2n}$, $k \in \{0, \dots, n-1\}$, of degree $n-1$ on $\mathbb{R}^{2n}$ are defined by

$$\varphi_k(x,u)(\xi_1, \dots, \xi_{n-1}) :=$$

$$\frac{1}{k!(n-1-k)!\omega_{n-k}} \sum_{\sigma \in S(n-1)} \operatorname{sgn}(\sigma) \left\langle \bigwedge_{i=1}^{k} \Pi_1 \xi_{\sigma(i)} \wedge \bigwedge_{i=k+1}^{n-1} \Pi_2 \xi_{\sigma(i)} \wedge u, \Omega_n \right\rangle,$$

where $\xi_1, \dots, \xi_{n-1} \in \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$.
A straightforward calculation shows that

$$\langle a_K(x,u), \varphi_k(x,u) \rangle = \frac{1}{\omega_{n-k}} \sum_{|I|=n-1-k} \frac{\prod_{i \in I} k_i(x,u)}{\prod_{i=1}^{n-1} \sqrt{1 + k_i(x,u)^2}}$$

for $\mathcal{H}^{n-1}$ almost all $(x,u) \in \operatorname{Nor} K$, hence

$$T_K(\mathbf{1}_\eta \varphi_k) = \Lambda_k(K, \eta).$$

# Current representation of $\varphi_k^{r,s,1}$

Let $n \geq 3$, $k \in \{1, \ldots, n-2\}$ and $r, s \in \mathbb{N}_0$. Let $(x, u) \in \mathbb{R}^{2n}$ and $\mathbf{v} = (v_1, \ldots, v_{r+s+2}) \in (\mathbb{R}^n)^{r+s+2}$. For $\xi_1, \ldots, \xi_{n-1} \in \mathbb{R}^{2n}$, define

$$\widetilde{\varphi}_k^{r,s}(x, u; \mathbf{v}; \xi_1, \ldots, \xi_{n-1}) := \frac{C_{n,k}^{r,s}}{(k-1)!(n-1-k)!} x^r(v_1, \ldots, v_r) u^s(v_{r+1}, \ldots, v_{r+s})$$

$$\times \sum_{\sigma \in S(n-1)} \operatorname{sgn}(\sigma) \left\langle v_{r+s+1}, \Pi_1 \xi_{\sigma(1)} \right\rangle \left\langle v_{r+s+2} \wedge \bigwedge_{i=2}^{k} \Pi_1 \xi_{\sigma(i)} \wedge \bigwedge_{i=k+1}^{n-1} \Pi_2 \xi_{\sigma(i)} \wedge u, \Omega_n \right\rangle .$$

For fixed $x, u, \mathbf{v}$, the map

$$\widetilde{\varphi}_k^{r,s}(x, u; \mathbf{v}; \cdot) : (\mathbb{R}^{2n})^{n-1} \to \mathbb{R}$$

is multilinear and alternating, hence an element of $\bigwedge^{n-1} \mathbb{R}^{2n}$.

# Current representation of $\varphi_k^{r,s,1}$

Symmetrization gives

$$\begin{aligned}&\varphi_k^{r,s}(x,u;\mathbf{v};\xi_1,\dots,\xi_{n-1})\\ &:=\frac{1}{(r+s+2)!}\sum_{\tau\in S(r+s+2)}\widetilde{\varphi}_k^{r,s}(x,u;v_{\tau(1)},\dots,v_{\tau(r+s+2)};\xi_1,\dots,\xi_{n-1}).\end{aligned}$$

Then

$$\varphi_k^{r,s}(x,u):(\xi_1,\dots,\xi_{n-1})\mapsto\varphi_k^{r,s}(x,u;\,\cdot\,;\xi_1,\dots,\xi_{n-1})$$

is an $(n-1)$-covector of $\mathbb{T}^{r+s+2}$, i.e. $\in\bigwedge^{n-1}(\mathbb{R}^{2n},\mathbb{T}^{r+s+2})$.
The map

$$\varphi_k^{r,s}:\mathbb{R}^{2n}\to\bigwedge^{n-1}(\mathbb{R}^{2n},\mathbb{T}^{r+s+2}),\qquad (x,u)\mapsto\varphi_k^{r,s}(x,u),$$

is a differential form of degree $n-1$ on $\mathbb{R}^{2n}$ with coefficients in $\mathbb{T}^{r+s+2}$, i.e. $\in\mathcal{E}^{n-1}(\mathbb{R}^{2n},\mathbb{T}^{r+s+2}):=\mathcal{E}(\mathbb{R}^{2n},\bigwedge^{n-1}(\mathbb{R}^{2n},\mathbb{T}^{r+s+2}))$.

# Properties

For $(x, u) \in \mathbb{R}^{2n}$, $a \in \bigwedge_{n-1} \mathbb{R}^{2n}$, $\vartheta \in \mathrm{O}(n)$:

- $\langle a, \varphi_k^{r,s}(x, u) \rangle \in \mathbb{T}^{r+s+2}$,
- $\langle \vartheta a, \varphi_k^{r,s}(\vartheta x, \vartheta u) \rangle = \vartheta \langle a, \varphi_k^{r,s}(x, u) \rangle$.

where $\vartheta\xi := (\vartheta p, \vartheta q)$ for $\xi = (p, q) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$.

The following lemma justifies these definitions.

**Lemma.** *If $P \in \mathcal{P}^n$ and $\eta \in \mathcal{B}(\Sigma^n)$, then*

$$T_P\left(\mathbf{1}_\eta \varphi_k^{r,s}\right) = \phi_k^{r,s,1}(P, \eta).$$

**Proof** For $P \in \mathcal{P}^n$ and $\eta \in \mathcal{B}(\Sigma^n)$, we show that

$$T_P\left(\mathbf{1}_\eta \widetilde{\varphi}_k^{r,s}(\cdot\,;\mathbf{v};\cdot)\right) = C_{n,k}^{r,s} \sum_{F \in \mathcal{F}_k(P)} Q_{L(F)}(v_{r+s+1}, v_{r+s+2}) \times \int_{\eta \cap (F \times \nu(P,F))} x^r(v_1,\dots,v_r) u^s(v_{r+1},\dots,v_{r+s})\, \mathcal{H}^{n-1}(\mathrm{d}(x,u)),$$

**Proof** For $P \in \mathcal{P}^n$ and $\eta \in \mathcal{B}(\Sigma^n)$, we show that

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For this, we start from the disjoint decomposition

$$\eta \cap \mathrm{Nor}\, P = \bigcup_{j=0}^{n-1} \bigcup_{F \in \mathcal{F}_j(P)} \eta \cap (\mathrm{relint}\, F \times \nu(P,F))$$

and use information about $k_i(x,u) \in \{0,\infty\}$ available for polytopes, when $(x,u) \in F \times \nu(P,F)$ for a $j$-face $F$ of $P$: exactly $j$ of the $k_i(x,u)$ are zero.

**Lemma**

(a) *For* $K \in \mathcal{K}^n$, $T_K$ *is a cycle.*

(b) *The map* $K \mapsto T_K$ *is a valuation on* $\mathcal{K}^n$.

(c) *If* $K_i, K \in \mathcal{K}^n$, $i \in \mathbb{N}$, *and* $K_i \to K$ *in the Hausdorff metric, as* $i \to \infty$, *then* $T_{K_i}(\varphi) \to T_K(\varphi)$ *for all differential forms* $\varphi$ *of degree* $n - 1$ *on* $\mathbb{R}^{2n}$.

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**Theorem** *The map $\mathcal{K}^n \times \mathcal{B}(\Sigma^n) \to \mathbb{T}^{r+s+2}$ defined by $(K, \eta) \mapsto T_K\left(\mathbf{1}_\eta \varphi_k^{r,s}\right)$, satisfies the properties (a) – (d). Hence*

$$\phi_k^{r,s,1}(K, \eta) := T_K\left(\mathbf{1}_\eta \varphi_k^{r,s}\right)$$

*defines a weakly continuous extension of the functional first introduced for polytopes.*

# Remarks

- $$\phi_k^{0,s,1}(P,\Sigma^n) = Q\Phi_k^{0,s}(P) - 2\pi(s+2)\,\Phi_k^{0,s+2}(P).$$
- More involved relations are available for $\phi_k^{r,s,1}$.
- We have
$$\phi_k^{r,s,1}(K,\eta) = C_{n,k}^{r,s}\int_{\eta\cap\operatorname{Nor}K} x^r u^s \sum_{i=1}^{n-1} b_i(x,u)^2 \sum_{\substack{|I|=n-1-k\\ i\notin I}} \frac{\prod_{j\in I} k_j(x,u)}{\mathbb{K}(x,u)}\,\mathcal{H}^{n-1}(\mathrm{d}(x,u)).$$

  This simplifies for $k=1,n-2$. If $n=3$, for instance,
$$\phi_1^{r,s,1}(K,\eta) = C_{3,1}^{r,s}\int_{\eta\cap\operatorname{Nor}K} x^r u^s \frac{k_1(x,u)b_2(x,u)^2 + k_2(x,u)b_1(x,u)^2}{\mathbb{K}(x,u)}\,\mathcal{H}^2(\mathrm{d}(x,u)).$$

  If $K$ is smooth, this simplifies further.

# The general classification result

**Theorem** *For $p \in \mathbb{N}_0$, let $T_p(\mathcal{K}^n)$ denote the real vector space of all mappings $\Gamma : \mathcal{K}^n \times \mathcal{B}(\Sigma^n) \to \mathbb{T}^p$ with the following properties.*

(a) $\Gamma(K, \cdot)$ *is a $\mathbb{T}^p$-valued measure, for each $K \in \mathcal{K}^n$;*

(b) $\Gamma$ *is isometry covariant;*

(c) $\Gamma$ *is locally defined;*

(d) *is weakly continuous.*

*Then a basis of $T_p(\mathcal{K}^n)$ is given by the mappings $Q^m \phi_k^{r,s,j}$, where $m, r, s \in \mathbb{N}_0$ and $j \in \{0, 1\}$ satisfy $2m + 2j + r + s = p$ and where $k \in \{0, \ldots, n-1\}$, but $j = 0$ if $k \in \{0, n-1\}$.*

# The general classification result

The preceding theorem is essentially equivalent to (follows from) the next theorem.

**Theorem** *Let $p \in \mathbb{N}_0$. Let $\Gamma : \mathcal{K}^n \times \mathcal{B}(\Sigma^n) \to \mathbb{T}^p$ be a mapping with the following properties.*

(a) *$\Gamma(K, \cdot)$ is a $\mathbb{T}^p$-valued measure, for each $K \in \mathcal{K}^n$;*

(b) *$\Gamma$ is translation invariant and rotation covariant;*

(c) *$\Gamma$ is locally defined;*

(d) *is weakly continuous.*

*Then $\Gamma$ is a linear combination, with constant coefficients, of the mappings $Q^m \phi_k^{0,s,j}$, where $m, s \in \mathbb{N}_0$ and $j \in \{0, 1\}$ satisfy $2m + 2j + s = p$ and where $k \in \{0, \dots, n-1\}$, but $j = 0$ if $k \in \{0, n-1\}$.*

# Ideas of Proof

- Reduction to fixed degreee of homogeneity $k \in \{1, \dots, n-2\}$ and $n \geq 3$.
- On polytopes $P$ the mapping $\Gamma$ is of the form

$$\Gamma(P, \cdot) = \sum_{\substack{m,j,s \geq 0 \\ 2m+2j+s=p}} c_{mjs} Q^m \phi_k^{0,s,j}(P, \cdot)$$

with constants $c_{mjs}$. $\Gamma$ and the mappings $\phi_k^{0,s,0}$ and $\phi_k^{0,s,1}$ are weakly continuous. The mapping $\Gamma'$ defined by

$$\Gamma' := \Gamma - \sum_{\substack{m,j,s \geq 0,\, j \leq 1 \\ 2m+2j+s=p}} c_{mjs} Q^m \phi_k^{0,s,j}$$

satisfies (a) – (d), on polytopes $P$ it is of the form

$$\Gamma'(P, \cdot) = \sum_{\substack{m,s \geq 0,\, j \geq 2 \\ 2m+2j+s=p}} c_{mjs} Q^m \phi_k^{0,s,j}(P, \cdot).$$

**Show that the constants $c_{mjs}$ with $j \geq 2$ are zero.**

# How to show that $c_{mjs} = 0$ for $j \geq 2$.

- Let $K_h$ be a cap of height $h$ cut off from a paraboloid of revolution with axis $\mathbb{R}e_n$.

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- Let $K_h$ be a cap of height $h$ cut off from a paraboloid of revolution with axis $\mathbb{R}e_n$.
- Clearly, $K_h$ has plenty of rotational symmetries.

# How to show that $c_{mjs} = 0$ for $j \geq 2$.

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- Clearly, $K_h$ has plenty of rotational symmetries.
- Approximate $K_h$ by a sequence of inscribed polytopes with limited symmetries:
  - choose a cubical grid of width $2t$, $t > 0$, in $\mathbb{R}^{n-1}$ and lift its vertices to the paraboloid;
  - let $P_{t,h}$ be the convex hull of these lifted points;
  - note that $P_{t,h} \to K_h$ as $t \to 0+$.

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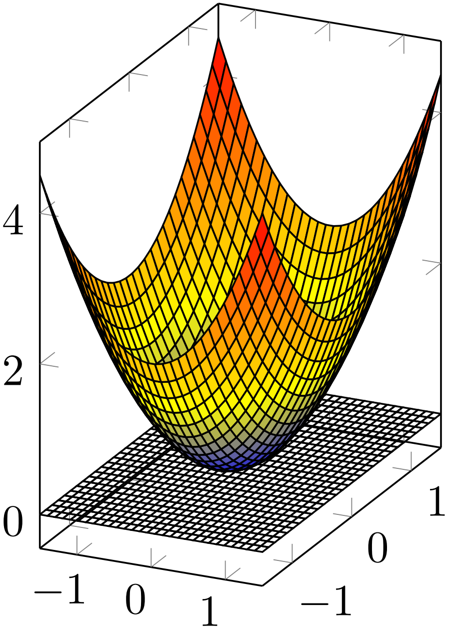
  The symmetries of $P_t$ can be controlled.
- We show that the covariance properties of $\Gamma'(P_{t,h}, \cdot)$ are restricted so strongly that this can be extended to the limiting case. This finally leads to a contradiction with the covariance properties of $\Gamma'(K_h, \cdot)$ if $\Gamma' \not\equiv 0$.


4
2
0
−1
0
1
−1
0
1

The remaining part of the proof requires a combination of (elementary) algebraic and geometric considerations as well as approximation arguments. We write again $\Gamma$ instead of $\Gamma'$ and

$$\Gamma(K, f) := \int_{\Sigma^n} f(u)\,\Gamma(K, \mathrm{d}(x, u))$$

for $K \in \mathcal{K}^n$ and a continuous real function $f$ on $\mathbb{S}^{n-1}$. For a polytope $P \in \mathcal{P}^n$, we define

$$W_k(P, f) := \sum_{F \in \mathcal{F}_k(P)} \mathcal{H}^k(F) \int_{\nu(P,F)} f \,\mathrm{d}\mathcal{H}^{n-k-1}.$$

**Lemma** *Let $s_0$ be the smallest number $s$ for which $c_{mjs} \neq 0$ for some $m, j$, set $q := (p - s_0)/2$ and $c_j := ((2q)!s_0!/p!)c_{(q-j)js_0} C_{n,k}^{0,s_0}$. Let $d$ be the largest $j \in \{2, \dots, q\}$ for which $c_j \neq 0$. Let $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$ and $0 < \varepsilon < 1$ be such that $\omega$ is $\varepsilon$-close to $-e_n$. Let $f$ be a nonnegative, continuous real function on $\mathbb{S}^{n-1}$ with support in $\omega$. For $P \in \mathcal{P}^n$, define*

$$\Delta(P, f) := \sum_{j=2}^{d} c_j Q^{q-j} \sum_{F \in \mathcal{F}_k(P)} Q^j_{L(F)} \mathcal{H}^k(F) \int_{\nu(P,F)} f \, \mathrm{d}\mathcal{H}^{n-k-1} \in \mathbb{T}^{2q}.$$

*Let* $E' := (\underbrace{a, \dots, a}_{2q})$ *with* $a \in \mathbb{R}^{n-1}, \|a\| = 1$ *and*

$$E := (b_1, \dots, b_p) := (\underbrace{a, \dots, a}_{2q}, \underbrace{-e_n, \dots, -e_n}_{s_0}).$$

*Then*

$$\left|\Gamma(P, f)(E) - \Delta(P, f)(E')\right| \leq C_3 W_k(P, f)\varepsilon$$

*with a constant $C_3$ depending only on $\Gamma$.*

We apply this approximation to the polytopes $P_{t,h}$ to get

$$\begin{aligned}
&\Gamma(P_{h,t}, f)(E) \\
&= \sum_{j=2}^{d} c_j \sum_{i=1}^{b(n,k)} \sum_{F \in \mathcal{F}^i_{k,h}(P_{h,t})} (Q^{q-j} Q^j_{L_i})(E') \mathcal{H}^k(F) \int_{\nu(P_{h,t},F)} f \, \mathrm{d}\mathcal{H}^{n-k-1} + R_5(E) \\
&= b(n,k)^{-1} W_k(P_{h,t}, f) \left( \sum_{j=2}^{d} c_j Q^{q-j} \sum_{i=1}^{b(n,k)} Q^j_{L_i} \right)(E') + R_5(E)
\end{aligned}$$

with

$$|R_5(E)| \leq C_5 W_k(P_{h,t}, f)\varepsilon,$$

where $C_5$ depends only on $\Gamma$.

**This approximation is crucial for the next lemma.**

**Lemma** *Let $n \geq 4$. Under the assumptions made above, there exist a convex body $K \in \mathcal{K}^n$, a continuous function $f$ on $\mathbb{S}^{n-1}$, a $p$-tuple $E$, and a rotation $\vartheta \in \mathrm{O}(n)$ such that $K$ and $f$ are invariant under $\vartheta$, but $\Gamma(K, f)(\vartheta E) \neq \Gamma(K, f)(E)$.*

The proof of this lemma involves the study of the polynomials

$$\sum_{j=2}^{d} c_j (x_1^2 + \cdots + x_n^2)^{q-j} \sum_{I \subset \{1,\ldots,n\}, |I|=k} \left( \sum_{i \in I} x_i^2 \right)^j .$$

It is tempting to believe that $c_2 = \ldots = c_d = 0$ if this polynomial is rotation invariant. However, this is not true (not even for $n = 2$) and this is precisely the reason why several cases have to be distinguished and additional geometric arguments are required.

If this is proved, then the invariance of $K$ and $f$ under $\vartheta$ and the rotation covariance of $\Gamma$ give

$$\Gamma(K, f)(\vartheta E) = \Gamma(\vartheta K, \vartheta f)(\vartheta E) = \Gamma(K, f)(E).$$

This is a contradiction, which finishes the proof of Theorem 5.1 for $n \geq 4$.

For $n = 3$ an thus $k = 1$ (and $d$ odd) we work with a more flexible (depending on $d$) triangular complex in $\mathbb{R}^2$ and with Minkowski averages of the polytopes obtained by lifting the vertices of this complex.