Voronoi-based estimation of Minkowski tensors from digital images

Anne Marie Svane
Joint work with Daniel Hug and Markus Kiderlen

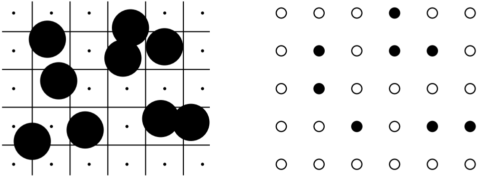
Aarhus University
CSGB Center for Stochastic Geometry and Advanced Bioimaging

Sandbjerg, September 26, 2014

Digital images

Suppose we study an object $X \subseteq \mathbb{R}^d$. (For instance via a microscope or scanner.)

The only information we have about X is a (black-and-white) digital image:

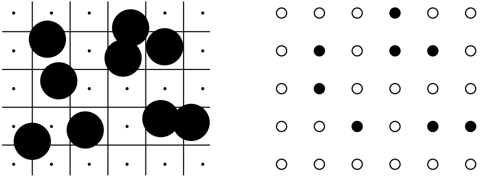


The pixel midpoints form a lattice $\mathbb{L}$. ($\mathbb{L} = \mathbb{Z}^d$)

Digital images

Suppose we study an object $X \subseteq \mathbb{R}^d$. (For instance via a microscope or scanner.)

The only information we have about X is a (black-and-white) digital image:



The pixel midpoints form a lattice $\mathbb{L}$. ($\mathbb{L} = \mathbb{Z}^d$)

Mathematically, the information in a black-and-white image is the set

$$X \cap \mathbb{L}$$

of black pixel midpoints.

Digital stereology

From the information

$$X \cap \mathbb{L},$$

we want to derive information about the geometry of X.

Digital stereology

From the information

$$X \cap \mathbb{L},$$

we want to derive information about the geometry of X.

In this talk: want to estimate the *Minkowski tensors*.

Digital stereology

From the information

$$X \cap \mathbb{L},$$

we want to derive information about the geometry of X.

In this talk: want to estimate the *Minkowski tensors*.

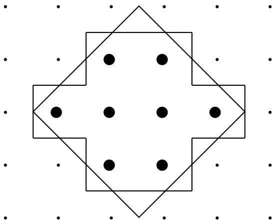
Change of resolution corresponds to scaling $\mathbb{L}$ by some $a > 0$. We then have the information

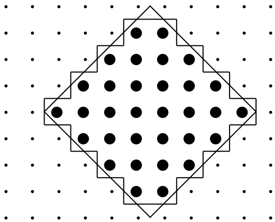
$$X \cap a\mathbb{L}.$$

Want estimators to converge when $a \to 0$.

Naive approach

Approximate X by the union of black pixels.





Good approximation of volume.

Boundary approximation is generally poor.
The boundary length is approximately $\sqrt{2}$ times too large.

Conditions on the object

We assume that our object $X \subseteq \mathbb{R}^d$

- is compact.
- is topologically regular, i.e. $X = \overline{\text{int}(X)}$.
- has positive reach, i.e. $\text{Reach}(X) > 0$.

Definition

Let $\text{Reach}(X)$ *be the largest number such that all* $x \in \mathbb{R}^d$ *with* $d(X, x) < \text{Reach}(X)$ *has a unique closest point in* X*.*

Convex sets and C^2 manifolds have positive reach.

Minkowski volume tensors

For $r, s \geq 0$, define the r-tensor:

$$\Phi_d^{r,0}(X) = \frac{1}{r!} \int_X x^r \, dx$$

In particular, $\Phi_d^{0,0}(X)$ is volume.

Minkowski volume tensors

For $r, s \geq 0$, define the r-tensor:

$$\Phi_d^{r,0}(X) = \frac{1}{r!} \int_X x^r \, dx$$

In particular, $\Phi_d^{0,0}(X)$ is volume.

Can estimate this by a Riemann sum

$$\Phi_d^{r,0}(X) \approx a^d c_{\mathbb{L}} \frac{1}{r!} \sum_{x \in X \cap a\mathbb{L}} x^r$$

where $c_{\mathbb{L}}$ is the volume of a lattice cell in $\mathbb{L}$.

General Minkowski tensors

For $r, s \geq 0$ and $k = 0, \ldots, d-1$, define

$$\Phi_k^{r,s}(X) = c_{r,s,k} \int_\Sigma x^r u^s C_k(X; d(x,u))$$

where:

$\Sigma = \mathbb{R}^d \times S^{d-1}$,
$C_k(X; \cdot)$ is the k'th generalized curvature measure on Σ,
$x^r u^s$ means the symmetric tensor product.

Again, $\Phi_k^{0,0}(X)$ is the kth the intrinsic volume.

The Steiner formula

Let $0 \geq R$ and define the parallel set

$$X^R = \{x \in \mathbb{R}^d \mid d(X, x) \leq R\}.$$

The Steiner formula

Let $0 \geq R$ and define the parallel set

$$X^R = \{x \in \mathbb{R}^d \mid d(X, x) \leq R\}.$$

Define the Voronoi tensor

$$\mathcal{V}_R^{r,s}(X) = \int_{X^R} p_X(x)^r (x - p_X(x))^s dx.$$

$\mathcal{V}_R^{0,2}(X)$ is the (total) Voronoi covariance measure of *Mérigot et al.* (2010).

The Steiner formula

Let $0 \geq R$ and define the parallel set

$$X^R = \{x \in \mathbb{R}^d \mid d(X, x) \leq R\}.$$

Define the Voronoi tensor

$$\mathcal{V}_R^{r,s}(X) = \int_{X^R} p_X(x)^r (x - p_X(x))^s dx.$$

$\mathcal{V}_R^{0,2}(X)$ is the (total) Voronoi covariance measure of *Mérigot et al.* (2010).

When $R < Reach(X)$, the (generalized) Steiner formula yields

$$\mathcal{V}_R^{r,s}(X) = c_{r,s} \sum_{k=0}^{d} \kappa_{k+s} R^{s+k} \Phi_{d-k}^{r,s}(X). \tag{1}$$

κ_k is the volume of the unit ball in $\mathbb{R}^k$.

The Steiner formula

Let $0 \geq R$ and define the parallel set

$$X^R = \{x \in \mathbb{R}^d \mid d(X, x) \leq R\}.$$

Define the Voronoi tensor

$$\mathcal{V}_R^{r,s}(X) = \int_{X^R} p_X(x)^r (x - p_X(x))^s dx.$$

$\mathcal{V}_R^{0,2}(X)$ is the (total) Voronoi covariance measure of *Mérigot et al.* (2010).

When $R < Reach(X)$, the (generalized) Steiner formula yields

$$\mathcal{V}_R^{r,s}(X) = c_{r,s} \sum_{k=0}^{d} \kappa_{k+s} R^{s+k} \Phi_{d-k}^{r,s}(X). \tag{1}$$

κ_k is the volume of the unit ball in $\mathbb{R}^k$.
For $r = s = 0$, this is the usual Steiner formula.

Idea of estimation

Choose $0 < R_0 < \cdots < R_d < \mathrm{Reach}(X)$.

Write the equations (1) in matrix form:

$$\begin{pmatrix} \mathcal{V}_{R_0}^{r,s}(X) \\ \vdots \\ \mathcal{V}_{R_d}^{r,s}(X) \end{pmatrix} = c_{r,s} \begin{pmatrix} \kappa_s R_0^s & \dots & \kappa_{s+d} R_0^{s+d} \\ \vdots & & \vdots \\ \kappa_s R_d^s & \dots & \kappa_{s+d} R_d^{s+d} \end{pmatrix} \begin{pmatrix} \Phi_d^{r,s}(X) \\ \vdots \\ \Phi_0^{r,s}(X) \end{pmatrix}$$

Idea of estimation

Choose $0 < R_0 < \cdots < R_d < \mathrm{Reach}(X)$.

Write the equations (1) in matrix form:

$$\begin{pmatrix} \mathcal{V}^{r,s}_{R_0}(X) \\ \vdots \\ \mathcal{V}^{r,s}_{R_d}(X) \end{pmatrix} = c_{r,s} \begin{pmatrix} \kappa_s R_0^s & \dots & \kappa_{s+d} R_0^{s+d} \\ \vdots & & \vdots \\ \kappa_s R_d^s & \dots & \kappa_{s+d} R_d^{s+d} \end{pmatrix} \begin{pmatrix} \Phi^{r,s}_d(X) \\ \vdots \\ \Phi^{r,s}_0(X) \end{pmatrix}$$

Can solve for the Minkowski tensors:

$$\begin{pmatrix} \Phi^{r,s}_d(X) \\ \vdots \\ \Phi^{r,s}_0(X) \end{pmatrix} = \frac{1}{c_{r,s}} \begin{pmatrix} \kappa_s R_0^s & \dots & \kappa_{s+d} R_0^{s+d} \\ \vdots & & \vdots \\ \kappa_s R_d^s & \dots & \kappa_{s+d} R_d^{s+d} \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{V}^{r,s}_{R_0}(X) \\ \vdots \\ \mathcal{V}^{r,s}_{R_d}(X) \end{pmatrix}$$

Idea of estimation

Choose $0 < R_0 < \cdots < R_d < \mathrm{Reach}(X)$.

Write the equations (1) in matrix form:

$$\begin{pmatrix} \mathcal{V}^{r,s}_{R_0}(X) \\ \vdots \\ \mathcal{V}^{r,s}_{R_d}(X) \end{pmatrix} = c_{r,s} \begin{pmatrix} \kappa_s R_0^s & \dots & \kappa_{s+d} R_0^{s+d} \\ \vdots & & \vdots \\ \kappa_s R_d^s & \dots & \kappa_{s+d} R_d^{s+d} \end{pmatrix} \begin{pmatrix} \Phi^{r,s}_d(X) \\ \vdots \\ \Phi^{r,s}_0(X) \end{pmatrix}$$

Can solve for the Minkowski tensors:

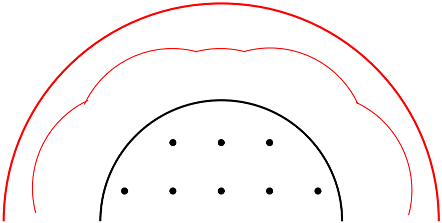
$$\begin{pmatrix} \Phi^{r,s}_d(X) \\ \vdots \\ \Phi^{r,s}_0(X) \end{pmatrix} = \frac{1}{c_{r,s}} \begin{pmatrix} \kappa_s R_0^s & \dots & \kappa_{s+d} R_0^{s+d} \\ \vdots & & \vdots \\ \kappa_s R_d^s & \dots & \kappa_{s+d} R_d^{s+d} \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{V}^{r,s}_{R_0}(X) \\ \vdots \\ \mathcal{V}^{r,s}_{R_d}(X) \end{pmatrix}$$

Idea: approximate $\mathcal{V}^{r,s}_{R_i}(X)$ by $\mathcal{V}^{r,s}_{R_i}(X \cap a\mathbb{L})$.

The algorithm

By definition

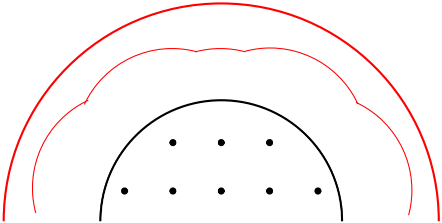
$$\mathcal{V}_R^{r,s}(X \cap a\mathbb{L}) = \int_{(X\cap a\mathbb{L})^R} p_{X\cap a\mathbb{L}}(y)^r (y - p_{X\cap a\mathbb{L}}(y))^s dy.$$



The algorithm

By definition

$$\mathcal{V}_R^{r,s}(X \cap a\mathbb{L}) = \int_{(X\cap a\mathbb{L})^R} p_{X\cap a\mathbb{L}}(y)^r (y - p_{X\cap a\mathbb{L}}(y))^s dy.$$



For intrinsic volumes:

$$\mathcal{V}_R^{0,0}(X) = V_d(X^R),$$
$$\mathcal{V}_R^{0,0}(X \cap a\mathbb{L}) = V_d((X \cap a\mathbb{L})^R).$$

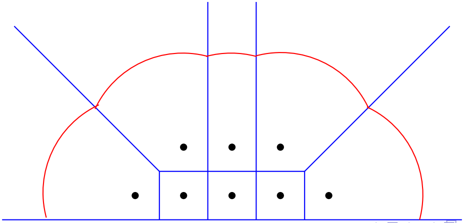
Voronoi expression

For each $x \in X \cap a\mathbb{L}$, define the Voronoi cell of x by

$$V_x = \{y \in \mathbb{R}^d \mid \forall z \in X \cap a\mathbb{L} : |y - x| \leq |y - z|\}.$$

Then $(X \cap a\mathbb{L})^R = \bigcup_{x \in X \cap a\mathbb{L}} V_x \cap B(x, R)$ and hence

$$\begin{aligned} \mathcal{V}_R^{r,s}(X \cap a\mathbb{L}) &= \sum_{x \in X \cap a\mathbb{L}} \int_{V_x \cap B(x,R)} x^r (y-x)^s dy \\ &= \sum_{x \in X \cap a\mathbb{L}} x^r \int_{(V_x - x) \cap B(0,R)} z^s dz. \end{aligned}$$



Convergence

Would like our estimators to converge to the true value when $a \to 0$.

Recall that:

$$\begin{pmatrix} \Phi_d^{r,s}(X) \\ \vdots \\ \Phi_0^{r,s}(X) \end{pmatrix} = \frac{1}{c_{r,s}} \begin{pmatrix} \kappa_s R_0^s & \dots & \kappa_{s+d} R_0^{s+d} \\ \vdots & & \vdots \\ \kappa_s R_d^s & \dots & \kappa_{s+d} R_d^{s+d} \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{V}_{R_0}^{r,s}(X) \\ \vdots \\ \mathcal{V}_{R_d}^{r,s}(X) \end{pmatrix}$$

So it is enough that

$$\lim_{a \to 0} \mathcal{V}_R^{r,s}(X \cap a\mathbb{L}) = \mathcal{V}_R^{r,s}(X).$$

Convergence - intrinsic volumes

For $X_1, X_2 \subseteq \mathbb{R}^d$ compact, define the Hausdorff distance

$$d_H(X_1, X_2) = \inf\{\varepsilon > 0 \mid X_1 \subseteq X_2^\varepsilon, X_2 \subseteq X_1^\varepsilon\}.$$

Theorem (Chazal, Cohen-Steiner, Mérigot (2010))

Let $X_1, X_2 \subseteq \mathbb{R}^d$ compact. Let $d_H(X_1, X_2) < \frac{R}{2}$. Then

$$|V_d(X_1^R) - V_d(X_2^R)| \leq C(d, X_1, R) d_H(X_1, X_2).$$

Convergence - intrinsic volumes

For $X_1, X_2 \subseteq \mathbb{R}^d$ compact, define the Hausdorff distance

$$d_H(X_1, X_2) = \inf\{\varepsilon > 0 \mid X_1 \subseteq X_2^\varepsilon, X_2 \subseteq X_1^\varepsilon\}.$$

Theorem (Chazal, Cohen-Steiner, Mérigot (2010))

Let $X_1, X_2 \subseteq \mathbb{R}^d$ compact. Let $d_H(X_1, X_2) < \frac{R}{2}$. Then

$$|V_d(X_1^R) - V_d(X_2^R)| \leq C(d, X_1, R) d_H(X_1, X_2).$$

Since X is topologically regular, $\lim_{a \to 0} d_H(X, X \cap a\mathbb{L}) = 0$.

Convergence - intrinsic volumes

For $X_1, X_2 \subseteq \mathbb{R}^d$ compact, define the Hausdorff distance

$$d_H(X_1, X_2) = \inf\{\varepsilon > 0 \mid X_1 \subseteq X_2^\varepsilon, X_2 \subseteq X_1^\varepsilon\}.$$

Theorem (Chazal, Cohen-Steiner, Mérigot (2010))

Let $X_1, X_2 \subseteq \mathbb{R}^d$ compact. Let $d_H(X_1, X_2) < \frac{R}{2}$. Then

$$|V_d(X_1^R) - V_d(X_2^R)| \leq C(d, X_1, R) d_H(X_1, X_2).$$

Since X is topologically regular, $\lim_{a\to 0} d_H(X, X \cap a\mathbb{L}) = 0$.

Corollary

$$\lim_{a\to 0} \mathcal{V}_R^{0,0}(X \cap a\mathbb{L}) = \mathcal{V}_R^{0,0}(X).$$

Convergence - general tensors

Theorem (Merigot et al. (2010), Hug, Kiderlen, S. (2014))

Let $X_1, X_2 \subseteq \mathbb{R}^d$ *be compact. Assume* $d_H(X_1, X_2) < \min\{\frac{R}{2}, diam(X_1), \frac{diam(X_1)^2}{R - d_H(X_1, X_2)}\}$*. Then*

$$|\mathcal{V}_R^{r,s}(X_1) - \mathcal{V}_R^{r,s}(X_2)| \leq C(d, r, s, X_1, R) d_H(X_1, X_2)^{\frac{1}{2}}.$$

Convergence - general tensors

Theorem (Merigot et al. (2010), Hug, Kiderlen, S. (2014))

Let $X_1, X_2 \subseteq \mathbb{R}^d$ *be compact. Assume* $d_H(X_1, X_2) < \min\{\frac{R}{2}, diam(X_1), \frac{diam(X_1)^2}{R - d_H(X_1, X_2)}\}$. *Then*

$$|\mathcal{V}_R^{r,s}(X_1) - \mathcal{V}_R^{r,s}(X_2)| \leq C(d, r, s, X_1, R) d_H(X_1, X_2)^{\frac{1}{2}}.$$

Corollary

$$\lim_{a \to 0} \mathcal{V}_R^{r,s}(X \cap a\mathbb{L}) = \mathcal{V}_R^{r,s}(X).$$

If X *is convex or a* C^2 *manifold, then the convergence speed is* $O(\sqrt{a})$.

A few words about the proof

It is enough to show that for a basis $e_1, \ldots, e_d \in \mathbb{R}^d$, the evaluations satisfy

$$|\mathcal{V}_R^{r,s}(X_1)(e_{i_1}, \ldots, e_{i_{r+s}}) - \mathcal{V}_R^{r,s}(X_2)(e_{i_1}, \ldots, e_{i_{r+s}})| \\ \leq \tilde{C}(d, r, s, X_1, R) d_H(X_1, X_2)^{\frac{1}{2}}.$$

A few words about the proof

It is enough to show that for a basis $e_1, \dots, e_d \in \mathbb{R}^d$, the evaluations satisfy

$$|\mathcal{V}_R^{r,s}(X_1)(e_{i_1}, \dots, e_{i_{r+s}}) - \mathcal{V}_R^{r,s}(X_2)(e_{i_1}, \dots, e_{i_{r+s}})| \\ \leq \tilde{C}(d, r, s, X_1, R) d_H(X_1, X_2)^{\frac{1}{2}}.$$

The left hand side can be written as

$$\left| \int_{X_1^R} f(p_{X_1}(x), x - p_{X_1}(x)) dx - \int_{X_2^R} f(p_{X_2}(x), x - p_{X_2}(x)) dx \right|.$$

where $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is some locally Lipshitz function.

The key ingredient is now a theorem by Chazal, Cohen-Steiner, and Mérigot:

$$\int_E |p_{X_1}(x) - p_{X_2}(x)| dx \leq C(d, X_1, E) d_H(X_1, X_2)^{\frac{1}{2}}.$$

Local tensors

We can consider the tensor valued measure, given on a Borel set $A \subseteq \mathbb{R}^d$ by

$$\Phi_k^{r,s}(X;A) = c_{r,s,k} \int_{A \times S^{d-1}} x^r u^s C_k(X; d(x,u)).$$

Define the Voronoi tensor measure

$$\mathcal{V}_R^{r,s}(X;A) = \int_{X^R} \mathbb{1}_A(p_X(x)) p_X(x)^r (x - p_X(x))^s dx.$$

Local tensors

We can consider the tensor valued measure, given on a Borel set $A \subseteq \mathbb{R}^d$ by

$$\Phi_k^{r,s}(X;A) = c_{r,s,k} \int_{A \times S^{d-1}} x^r u^s C_k(X; d(x,u)).$$

Define the Voronoi tensor measure

$$\mathcal{V}_R^{r,s}(X;A) = \int_{X^R} \mathbb{1}_A(p_X(x)) p_X(x)^r (x - p_X(x))^s dx.$$

Theorem

Let $X_i, X \subseteq \mathbb{R}^d$ be compact sets. If $\lim_{i\to\infty} d_H(X_i, X)$, then

$$\lim_{i\to\infty} \mathcal{V}_R^{r,s}(X_i;A) = \mathcal{V}_R^{r,s}(X;A)$$

for every Borel set A that satisfies $\mathcal{H}^d(p_X^{-1}(\partial A) \cap X^R) = 0$.

Local tensors

We can consider the tensor valued measure, given on a Borel set $A \subseteq \mathbb{R}^d$ by

$$\Phi_k^{r,s}(X;A) = c_{r,s,k} \int_{A \times S^{d-1}} x^r u^s C_k(X; d(x,u)).$$

Define the Voronoi tensor measure

$$\mathcal{V}_R^{r,s}(X;A) = \int_{X^R} \mathbb{1}_A(p_X(x)) p_X(x)^r (x - p_X(x))^s dx.$$

Theorem

Let $X_i, X \subseteq \mathbb{R}^d$ *be compact sets. If* $\lim_{i\to\infty} d_H(X_i, X)$*, then*

$$\lim_{i\to\infty} \mathcal{V}_R^{r,s}(X_i;A) = \mathcal{V}_R^{r,s}(X;A)$$

for every Borel set A *that satisfies* $\mathcal{H}^d(p_X^{-1}(\partial A) \cap X^R) = 0$.

Local tensors with $A \subseteq \Sigma$?

Discussion

Conclusions:

- Get algorithm for estimation of all Minkowski tensors.
- Proof of convergence when resolution goes to infinity.
- Simple expression in terms of Voronoi cells.
- Slower than local algorithms.
- Applies to other approximations of X than digital images.

Discussion

Conclusions:

- Get algorithm for estimation of all Minkowski tensors.
- Proof of convergence when resolution goes to infinity.
- Simple expression in terms of Voronoi cells.
- Slower than local algorithms.
- Applies to other approximations of X than digital images.

Future directions:

- Performance in practice?
- How to choose R_i?
- Extension to polyconvex sets?
- Extension to grey-valued images?